Transactions on Computational and Applied Mathematics (2025)
Clausius Scientific Press, Canada
DOI: 10.23977/tracam.2025.050103
ISSN 2616-1826 Vol. 5 Num. 1

Boundedness Theorem for Continuous Functions in the Perplex Number-plane

Zengrui Kang[1,*], Yuxun Zeng[2]

[1]*College of Sciences, Shihezi University, Shihezi, 832000, China*
[2]*College of Water Conservancy and Civil Engineering, South China Agricultural University, Guangzhou, Guangdong, 510642 China*
[*]*Corresponding author:17838887375@163.com*

Keywords: Perplex Number, Partial Order, Boundedness

Abstract: Continuous function theory is a very important part of functional theory. In this paper, this article research the properties of continuous functions in the P-plane, which are commutative rings containing zero factors generated by two real number. This article overcome the difficulties arising from the zero factorisation of P-numbers and prove that continuous functions in the P-plane are bounded by the decomposition of P-numbers and a geometric interpretation is given for the boundedness of continuous functions in the P-plane. For continuous Perplex functions that satisfy zero factor decomposition, based on the decomposition of Perplex numbers, we obtain the continuous boundedness theorem for Perplex functions. This will further provide a theoretical foundation and research impetus for P-analysis and give impetus to applications of the theory of continuous functions in the p-plane in physics.

1. Introduction

Richter has made significant contributions to the field by developing a geometric methodology for generalized complex and hyper-complex numbers. He proposed a technique for constructing multi-dimensional complex numbers using a general vector space structure and geometric multiplication rules. Richter extended the concepts of geometric vector product and exponential functions to higher dimensions, introducing new algebraic rules[1]. He also studied three-complex numbers, introduced geometric vector and spherical coordinate products, and demonstrated their equivalence. Richter derived Euler-type trigonometric representations and applied them to construct directional probability distributions in three-dimensional space, offering new insights into directional distribution theory[2]. Luna-Elizarraras systematically studied the integration theory for hyperbolic-valued functions, considering partial orders and hyperbolic intervals. She established foundational theorems for hyperbolic integration, providing a framework applicable to various mathematical and physical contexts involving hyperbolic numbers[3]. Richter generalized complex numbers to relate to semi-antinorms, ellipses, and matrix homogeneous functionals, introducing new classes and extending Euler's formula. He discussed solutions to quadratic equations and proved invariance properties of certain probability densities, broadening the framework for complex number theory and its applications in probability and statistics[4].

Ravasini studied uniformly continuous mappings on unbounded hyperbolic spaces, focusing on the space $C_w(x)$ of mappings with a modulus of continuity bounded by a concave function w. He proved that, in the sense of Baire categories, the modulus of continuity of a generic mapping in $C_w(x)$ is precisely w, providing insights into the structure of these mappings and their applications in functional analysis and related fields[5]. Lang discussed the classification of hyperbolic monopoles with continuous symmetries, presenting a framework that simplifies the construction of spherically symmetric hyperbolic monopoles. He derived a Structure Theorem, which provides new insights into the representation theory and has applications in understanding the mathematical structure of monopoles in Anti de-Sitter space and Skyrmions[6]. Griette, Magal, and Zhao studied the existence of traveling waves with continuous profiles for the hyperbolic Keller-Segel equation,focusing on cell-cell repulsion dynamics. They demonstrated the existence of such waves and applied their findings to wound healing processes, highlighting the importance of continuous and discontinuous wave profiles in biological contexts[7]. Belhamadia, Cassol, and Dubljevic developed a hyperbolic heat diffusion model to simulate finite speed of heat propagation in phase change problems, specifically applied to steel continuous casting. They demonstrated significant differences between parabolic and hyperbolic approaches, highlighting the model's effectiveness in capturing initial thermal dynamics and solid-liquid interface behavior[8].

Aigner et al. explored the exceptional characteristics and potential uses of hyperbolic metamaterials, noting their capacity to support waves with exceptionally large wavevectors and a high density of states. They examined a variety of structures and natural materials that display hyperbolic dispersion, highlighting their promise in boosting light-matter interactions and surpassing optical diffraction limits [9].Zhou, Zhang, and Yi studied the enhancement of near-field radiative heat transfer using hyperbolic metasurfaces crafted from uniaxial hyperbolic substrates. They showed that optimizing substrate parameters can substantially boost heat transfer efficiency, with potential applications in energy harvesting, thermal imaging, and radiative cooling [10].Smith et al. investigated the arithmetic properties of finite volume complex hyperbolic n-manifolds that contain infinitely many maximal properly immersed totally geodesic submanifolds. They established a superrigidity theorem for certain representations of complex hyperbolic lattices. Their findings have multiple applications, such as demonstrating the nonexistence of specific maps between complex hyperbolic manifolds and supporting Klingler's conjecture [11].Bensad and Ikemakhen devised a method to construct barycentric coordinates on the hyperbolic plane for arbitrary hyperbolic polygons. Using hyperbolic gnomonic projection, they derived coordinates analogous to those in the planar case. Their method also applies to the Poincaré disk model, with implications for hyperbolic mesh parameterization, deformation, and shape morphing [12].

Recent studies have made significant advancements in various aspects of hyperbolic numbers, including geometric approaches, integration theories, and applications in physics and materials science, providing a solid foundation for further exploration of their continuity and properties. Building on these developments, this paper makes a unique contribution by delving into the continuity theory of Perplex functions. Specifically, we focus on continuous Perplex functions that satisfy zero factor decomposition and derive a boundedness theorem for these functions. This work not only strengthens the theoretical framework of continuous function theory but also opens new avenues for exploring models in four-dimensional space and other advanced mathematical domains.

2. Preliminaries

In this paper, we will extend the function defined on the domain of real numbers to the function defined on the ring of p-numbers and continue to explore its boundedness.

The following is the definition of the set of perplex number.

$$\mathbb{P}=\left\{\xi=a+bh:a,b\in\mathbb{R}\right\}. \tag{1}$$

The perplex unit $\mathbb{P}$ satisfies $h^2=1$ and $k\neq\pm1$.

In this literature, h^+ and h^- are defined as two zero-divisiors in $\mathbb{P}$. The specific meanings and the reasons are as follows

$$h^+=\frac{(1+h)}{2},h^-=\frac{(1-h)}{2}, \tag{2}$$

$$(1+h)(1-h)=1-h^2=0. \tag{3}$$

It is noted that zero divisors are idempotent elements. They are also called as the idempoent of $\mathbb{P}$.

$$(h^-)^2=\frac{(1-h)}{2}\frac{(1-h)}{2}=h^-, \tag{4}$$

and we have

$$h^++h^-=\frac{(1+h)}{2}+\frac{(1-h)}{2}=1, \tag{5}$$

$$h^+-h^-=\frac{(1+h)}{2}-\frac{(1-h)}{2}=h, \tag{6}$$

$$h^+h^-=0. \tag{7}$$

Unlike the field $\mathbb{C}$, $\mathbb{P}$ is a commutative ring. It's operations of addition and multiplication are as follows

$$\xi_1+\xi_2=\left(a_1+b_1h\right)+\left(a_2+b_2h\right)=\left(a_1+a_2\right)+\left(b_1+b_2\right)h, \tag{8}$$

and

$$\xi_1\xi_2=\left(a_1+b_1h\right)\left(a_2+b_2h\right)=\left(a_1a_2+b_1b_2\right)+\left(a_1b_2+a_2b_1\right)h. \tag{9}$$

Where $\xi_1=a_1+b_1h$ and $\xi_2=a_2+b_2h$. For $\xi=a+bh$,this article define the real part of ξ as $\operatorname{Re}(\xi)$, which $\operatorname{Re}(\xi)=a$. We also define the perplex part of ξ as $\operatorname{Im}(\xi)$, which $\operatorname{Im}(\xi)=b$. This article denote the conjuate of ξ as $\bar{\xi}$ and $\bar{\xi}=a-bh$.

If $\xi=a+bh\in\mathbb{P}$, then the idempotent representation of it is

$$\xi=a+bh=(a+b)h^++(a-b)h^-=uh^++vh^-. \tag{10}$$

It's obvious that the real multiples of h^+ and h^- are the zero divisors. The reason is the multiples of h^+ and h^- are on the $y=x$ and $y=-x$ lines. So we have

$$\xi_1+\xi_2=(u_1+u_2)h^++(v_1+v_2)h^-, \tag{11}$$

$$\xi_1\xi_2=(u_1u_2)h^++(v_1v_2)h^-. \tag{12}$$

The set of non-negative hyperbolic numbers is

$$\mathbb{P}^+:=\left\{\xi=a+bh:a\geq0,b\geq0\right\}. \tag{13}$$

The set of non-positive hyperbolic numbers is

$$\mathbb{P}^- := \{\xi = a + bh : a \le 0, b \le 0\}. \quad (14)$$

Therefore, we define

$$\xi_1 \succeq \xi_2, \text{if} \xi_1 - \xi_2 \in \mathbb{P}^+. \quad (15)$$

and this article say ξ_1 is $\mathbb{P}$-greater than ξ_2
Similariy, we define

$$\xi_1 \preceq \xi_2, \text{if} \xi_1 - \xi_2 \in \mathbb{P}^-. \quad (16)$$

and this article say ξ_1 is $\mathbb{P}$-less than.

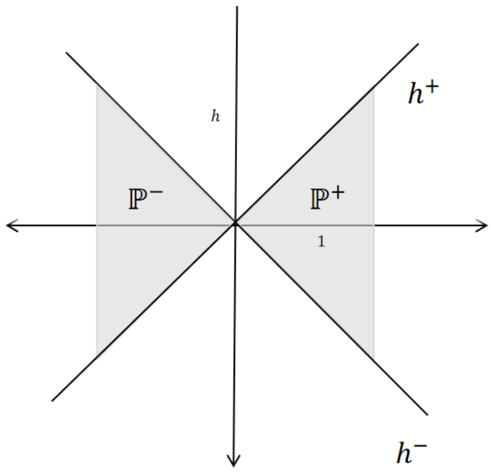

Figure 1. The concept of positive and negative perplex number

The relation $\preceq$ is a partial order in $\mathbb{P}$. It's transitive, reflexive and antisymmetric. We called $\xi_1 \preceq \xi_2$ if $a_1 \preceq a_2$, $b_1 \preceq b_2$. This article called $\xi_1 \succeq \xi_2$ iff $a_1 \succeq a_2, b_1 \succeq b_2$. Also, it's clear that $\xi \succeq 0$ is equvalent to $\xi \in \mathbb{P}^+$ and $\xi \succ 0$ is equivalent to $\xi \in \mathbb{P}^+ / \{0\}$. $\xi \preceq 0$ is equivalent to $\xi \in \mathbb{P}^-$ and $\xi \prec 0$ is equivalent to $\xi \in \mathbb{P}^- / \{0\}$. Figure.1 is the relation order. In the figure, x-axis and y-axis is a one-dimensional space. These spaces are embedded in $\mathbb{P}$. If the speed of light is taken as figure.1, we called the set of zero divisors with a positive real Part as the future direction, and we called the set of zero divisors with a negative real part as the past direction. Additionly, the points on the $y = x$ and $y = -x$ lines, the ones with $x > 0$ are the future direction, the ones with $x < 0$ are the past direction. The real line is embedded in $\mathbb{P}$ through the injectin $\tau : \mathbb{R} \to \mathbb{P}$, for all $a \in \mathbb{R}$,

$$\tau(a) = a = ah^+ + ah^-. \quad (17)$$

Definition 1: For $a = a_1 + b_1 h \in \mathbb{P}, b = a_2 + b_2 h \in \mathbb{P}$ such that $a \le b$, this article define the closed hyperbolic interval $[a,b]_{\mathbb{P}}$ by

$$[a,b]_{\mathbb{P}} = \{u \in \mathbb{P} : a \le u \le b\}. \quad (18)$$

Equivalently, $u = u_1 h^+ + u_2 h^- \in [\xi_1, \xi_2]_{\mathbb{P}}$ iff

$$a_1 \le a \le a_2 \text{ and } b_1 \le b \le b_2. \quad (19)$$

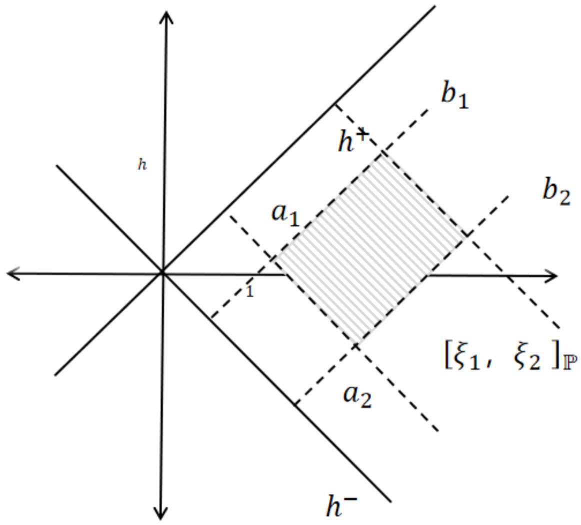

Figure 2. the concept of perplex number intervals

Figure 2 is the concept of perplex number intervals, which reveals the geometric significance of intervals $[\xi_1, \xi_2]_{\mathbb{P}}$. The hyperbolic interval $[a,b]_{\mathbb{P}}$ is degenerate, if $a-b$ is a non-negtive zero divisor hyperbolic number. And the hyperbolic interval $[a,b]_{\mathbb{P}}$ is non-degenerate, if $a-b$ is an inveitible positive hyperbolic number. The hyperbolic interbals have the notion of length as in the real interval. The length of any hyperbolic interval is a non-negative hyperbolic number.

3. Results

Definition 3.1 Continuity of points:

A sequence $\{P_n\}_{n\in\mathbb{N}}$ of perplex numbers $\mathbb{P}$-converges to the perplex number P_0, if for any strictly positive perplex number ε there exists $N \in \mathbb{N}$ such that there holds:

$$|P_n - P_0|_{\mathbb{P}} < \varepsilon \tag{20}$$

In this case this article say that the sequence $\{P_n\}_{n\in\mathbb{N}}$ converges to $\mathbb{P}_0$.

Using the idempotent representations

$$P_n = \alpha_{1n}h^+ + \alpha_{2n}h^-; P_0 = \alpha_{10}h^+ + \alpha_{20}h^- \tag{21}$$

This article obtain that, equivalently,

$$|\alpha_{1n} - \alpha_{10}| < \varepsilon_1 \text{ and } |\alpha_{2n} - \alpha_{20}| < \varepsilon_2 \tag{22}$$

Which means that the sequence $\{P_n\}_{n\in\mathbb{N}}$ converges to the perplex numbers P_0 with respect to the perplex-valued norm if and only if it converges to P_0 with respect to the Euclidean norm. Even though the two norms cannot be directly compared because they take values in different rings, they still yield the same sets of convergent and divergent sequences.

Definition 3.2 Continuity of a function at a point:

A proplex function is continuous at a point $P_0 \in \Omega \subset \mathbb{P}$, if $\lim\limits_{z\to z0} F(P)$ exists and

$$\lim_{z \to z_0} f(P) = f(P_0). \tag{23}$$

It means

$$\forall \varepsilon \in \mathbb{P}^+, \exists \delta \succ 0, \forall |p - p_0| < \delta, s.t. |f(p) - f(p_0)| \prec \varepsilon. \tag{24}$$

Theorem3.1(Boundedness of $\mathbb{P}$-variable functions): If the function $f(p)$ is continuous on the closed interval $[a,b]_{\mathbb{P}}$, and $f(p) = f_1(u)h^+ + f_2(v)h^-$, then $f(p)$ is bounded on the closed interval $[a,b]_{\mathbb{P}}$, as shown in Figure 3.

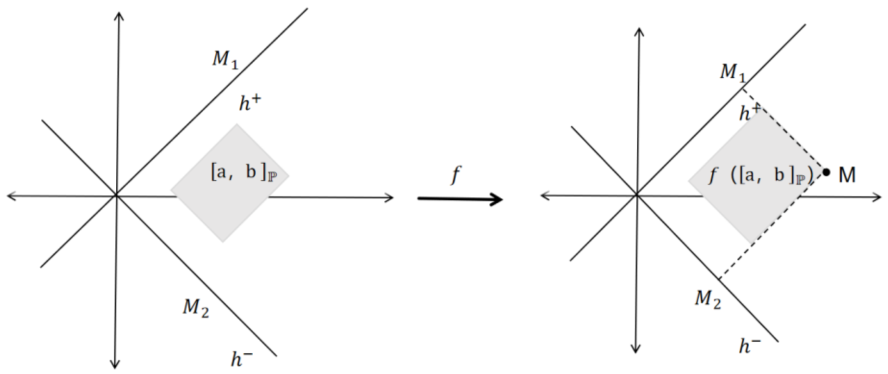

Figure 3. The concept of Boundedness Theorem

Proof: Based on the continuity of the $\mathbb{P}$-variable function $f(p)$ on a interval $[a,b]_{\mathbb{P}}$, it has $\forall \varepsilon = \varepsilon_1 h^+ + \varepsilon_2 h^- \succ 0, \exists \delta = \delta_1 h^+ + \delta_2 h^- \succ 0, for \forall h = uh^+ + vh^- \in [a,b]_{\mathbb{P}}$, has $|f(p) - f(p_0)|_{\mathbb{P}} \prec \varepsilon,$ which $p_0 = u_0 h^+ + v_0 h^-$ is a certain point in interval $[a,b]_{\mathbb{P}}$ and $a = a_1 h^+ + a_2 h^-, a = b_1 h^+ + b_2 h^-$.

Then according to the Decomposability of function $f(p)$, it has

$$\begin{aligned} |f(p) - f(p_0)| &= \left| \left(f_1(u)h^+ + f_2(v)h^- \right) - \left(f_1(u_0)h^+ + f_2(v_0)h^- \right) \right|_{\mathbb{P}} \\ &= |f_1(u) - f_1(u_0)|h^+ + |f_1(v) - f_1(v_0)|h^- \prec \varepsilon = \varepsilon_1 h^+ + \varepsilon_2 h^-, \end{aligned} \tag{25}$$

Through simplify, it has

$$|f_1(u) - f_1(u_0)| \prec \varepsilon_1, |f_1(v) - f_1(v_0)| \prec \varepsilon_2. \tag{26}$$

Based on the boundedness of continuous functions on closed intervals according to real analysis, it has $|f_1(u)| \le m_1, |f_2(v)| \le m_2$ .So

$$\forall p \in [a,b]_{\mathbb{P}}, |f(p)|_{\mathbb{P}} = |f_1(u)|h^+ + |f_2(v)|h^- \le m_1 h^+ + m_2 h^- = M. \tag{27}$$

Thus, the conclusion is $f(p)$ is bounded in interval $[a,b]_{\mathbb{P}}$.

4. Conclusions

This study delves into the examination of continuous functions operating on the hyperbolic plane, which is defined by a hyperbolic interval originating from a pair of real numbers. Although such functions encounter complexities due to zero factorization, this paper effectively addresses these challenges and establishes that these functions on the hyperbolic plane are indeed bounded. The proof provided offers a geometric perspective to understand their bounded nature. This finding is pivotal as it not only forms the groundwork for the theory of p-analysis but also supplies substantial theoretical backing and momentum for the utilization of continuous functions across disciplines, particularly in physics.

References

[1] Richter W D. On hyperbolic complex numbers[J]. Applied Sciences, 2022, 12(12): 5844.

[2] Richter W D. Three-complex numbers and related algebraic structures[J]. Symmetry, 2021, 13(2): 342.

[3] Luna-Elizarrarás M E. Integration of functions of a hyperbolic variable[J]. Complex Analysis and Operator Theory, 2022, 16(3): 35.

[4] Richter W D. Complex numbers related to semi-antinorms, ellipses or matrix homogeneous functionals[J]. Axioms, 2021, 10(4): 340.

[5] Ravasini D. Generic uniformly continuous mappings on unbounded hyperbolic spaces[J]. Journal of Mathematical Analysis and Applications, 2024, 538(1): 128440.

[6] Lang C J. Hyperbolic monopoles with continuous symmetries[J]. Journal of Geometry and Physics, 2024, 203: 105258.

[7] Griette Q, Magal P, Zhao M. Traveling waves with continuous profile for hyperbolic Keller-Segel equation[J]. arXiv preprint arXiv:2204.06920, 2022.

[8] Belhamadia Y, Cassol G O, Dubljevic S. Numerical modelling of hyperbolic phase change problems: Application to continuous casting[J]. International Journal of Heat and Mass Transfer, 2023, 209: 124042.

[9] Aigner A, Dawes J M, Maier S A, et al. Nanophotonics shines light on hyperbolic metamaterials[J]. Light, Science & Applications, 2022, 11.

[10] Zhou C L, Zhang Y, Yi H L. Enhancement and manipulation of near-field thermal radiation using hybrid hyperbolic polaritons[J]. Langmuir, 2022, 38(25): 7689-7698.

[11] Bader U, Fisher D, Miller N, et al. Arithmeticity, superrigidity and totally geodesic submanifolds of complex hyperbolic manifolds[J]. Inventiones mathematicae, 2023, 233(1): 169-222.

[12] eddine Bensad A, Ikemakhen A. Hyperbolic barycentric coordinates and applications[J]. Computer Aided Geometric Design, 2022, 95: 102086.