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Research on the Control of Chaotic Systems Based on the Small Gain Theorem

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Abstract: This paper discusses the input-to-state stability (ISS) of nonlinear systems and applies the small gain theorem in the control of chaotic systems. Firstly, the concept of ISS is defined and provides a way to determine whether the system is ISS. Secondly, a feedback control method is designed using the small gain theorem to make the origin of the new chaotic system globally asymptotically stable. Finally, simulation results verify the effectiveness of the control method.

1. Introduction

Chaotic systems are widely present in nature [1] and engineering applications [2], but their unpredictability and complexity often pose challenges. By studying the control and synchronization of chaotic systems, we can improve our understanding of these systems and develop more controllable and predictable methods [3] - [5]. This helps scientists and engineers better utilize chaos phenomena to solve a range of practical problems, thereby improving the performance and stability of the system.

Chaotic systems are characterized by sensitivity to initial conditions, which is also common in many nonlinear dynamic systems. These systems exhibit unpredictable and complex behavior, making them crucial for research and control, particularly in fields such as engineering [6], biology [7], and economics [8]. As emphasized in [9], the inherent nonlinearity of chaotic systems requires the development of robust control strategies to ensure stability and the required performance.

In modern control theory, the Input State Stability (ISS) attribute provides a powerful framework for analyzing and designing control systems. The concept of ISS links the behavior of system states with input signals, ensuring robust performance even in the presence of interference. The small gain theorem is a key tool for determining the stability of interconnected systems, especially for chaos and nonlinear dynamics.

Recent studies [10-12] have explored various chaos control methods. These methods include feedback control and synchronization techniques aimed at stabilizing chaotic behavior and ensuring global asymptotic stability.

The small gain theorem plays a crucial role in chaos control by providing a framework for analyzing and stabilizing interconnected subsystems. This theorem ensures that if each subsystem is stable, the entire system can also be stable. This is crucial for controlling complex chaotic systems with multiple interacting components. It helps to design feedback controllers that can effectively suppress chaotic dynamics.

The structure of this article is as follows. Section 2 introduces chaotic systems and outlines the problem of controlling chaotic behavior in nonlinear systems. In Section 3, we reviewed key definitions and theorems, including Input State Stability (ISS) and Small Gain Theorem, which are crucial for understanding the stability conditions of chaotic systems. In Sections 4 and 5, we applied the small gain theorem to design a feedback controller for chaos control, ensuring global asymptotic stability. Finally, Section 6 introduced numerical simulations to verify the effectiveness of the proposed control strategy in stabilizing chaotic systems.

2. Manuscript Preparation

Chaos control is an important problem in nonlinear dynamics and control theory, where the goal is to stabilize unstable equilibrium points in a chaotic system. Chaotic systems exhibit sensitive dependence on initial conditions and unpredictable long-term behavior. The design of effective control strategies for such systems is challenging due to their inherent nonlinearities and high sensitivity.

The primary problem addressed in this paper is how to use the small-gain theorem to design a chaos control method that ensures the global asymptotic stability of the origin for a given chaotic system. This involves leveraging ISS properties and feedback control mechanisms to achieve stabilization.

3. Basic Definitions

Definition 1 Consider the following nonlinear system:

$$\dot{x} = f(x,u) \tag{1}$$

When state $x \in R^n$, input $u \in R^m$, in which $f(0,0)=0$, and $f(x,u)$ locally satisfies Lipschitz condition on $R^n \times R^m$. The input of the system (1): $u:[0,\infty) \to R^m$ is a piecewise continuous bounded function, and the norm $\|u(\cdot)\|_\infty = \sup_{t\geq 0}\|u(t)\|$ is defined on the set of all piecewise continuous bounded functions, resulting in the model space which is denoted as L_∞^m. If there exists a KL function Equation $\beta(\cdot,\cdot)$ and K-Class function $\gamma(\cdot)$, for any input $u(t) \in L_\infty^m$ and initial state $x^0 \in R^n$, the following holds:

$$\|x(t,x^0)\| \leq \beta(\|x^0\|,t) + \gamma(\|u(\cdot)\|_\infty)\,, t>0\,;$$

Then the system (1) is Input-to-State Stable (ISS), and $\gamma(\cdot)$ is the gain function [7].

Definition 2 For the system (1), if for the C^1 function $V: R^n \to R$, there is a K_∞ function $\underline{\alpha}(\cdot)$、$\overline{\alpha}(\cdot)$、$\alpha(\cdot)$ and a K-Class function $\chi(\cdot)$ such that for all $x \in R^n$, then

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|) \tag{2}$$

and $\|x\| \geq \chi(\|u\|) \Rightarrow \dfrac{\partial V}{\partial x} f(x,u) \leq -\alpha(\|x\|)$ are exist, then V is the ISS-Lyapunov function [13].

Here are two theorems for determining whether a system is ISS:

Theorem 1 System (1) is ISS if and only if an ISS-Lyapunov function exists.

In this case, the gain function $\gamma(r)=\underline{\alpha}^{-1}\circ\overline{\alpha}\circ\chi(r)$ can be obtained.

Theorem 2 Consider system (1), for the C^1 function $V:R^n\rightarrow R$, if and only if K_∞ function $\underline{\alpha}(\cdot)$、$\overline{\alpha}(\cdot)$、$\alpha(\cdot)$ and a K-Class function $\sigma(\cdot)$ such that for all $x\in R^n$ and $u\in R^m$, function (2)and $\frac{\partial V}{\partial x}f(x,u)\leq-\alpha(\|x\|)+\sigma(\|u\|)$ exist, The function V is ISS-Lyapunov function.

The specific proof of this theorem can be found in reference [9].

4. Small gain theorem

Consider the system

$$\begin{cases}\dot{x}_1=f_1(x_1,x_2)\\ \dot{x}_2=f_2(x_1,x_2,u)\end{cases}\tag{3}$$

in which, $x_2\in R^{n_2}$ $u\in R^m$ and $f_1(0,0)=0$, $f_2(0,0,0)=0$ Supposed that the first subsystem, when x_1 is considered to be the state and x_2 is the input, is ISS and has a gain function $\gamma_1(\cdot)$; For the second subsystem, when x_2 it is considered to be the state and input x_1 and u, it is ISS and has a gain function $\gamma_2(\cdot)$.

The expression of the small gain theorem is as follows:

Theorem 3 If the condition $\gamma_1(\gamma_2(r))<r$ holds, for all $r>0$, system (3) is viewed as a system with state $x=(x_1,x_2)$ and input u, and it is ISS.

The specific proof of this theorem can be found in reference [13].

If the conditions of theorem 3 are satisfied, when u=0, then the origin of the system (3) is globally asymptotically stable.

Consider the following new chaotic system:

$$\begin{cases}\dot{x}=a(y-x)\\ \dot{y}=bx+cy-xz\\ \dot{z}=x^2-hz\end{cases}\tag{4}$$

When a=20,b=14,c=10.6,h=2.8, the system exhibits chaotic behavior and has three unstable equilibrium points: $E_1(0,0,0)$, $E_2(\sqrt{h(b+c)},\sqrt{h(b+c)},b+c)$, $E_3(-\sqrt{h(b+c)},-\sqrt{h(b+c)},b+c)$.

Take the initial value x(0)=y(0)=z(0)=20, and the chaotic attractor of the new system is shown in the figure1.

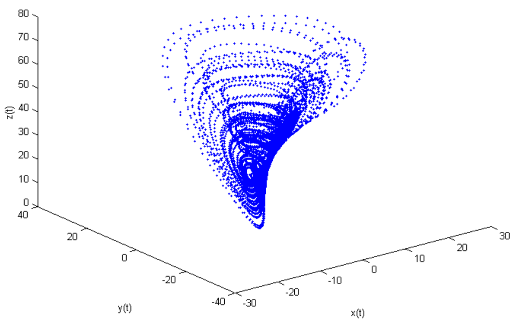

Figure 1: Chaotic attracto

5. Control Design

Using $E(\overline{x},\overline{y},\overline{z})$ to represent any points in E_1,E_2,E_3. The feedback control is designed below to make the points $E(\overline{x},\overline{y},\overline{z})$ globally asymptotically stable.

First, translate the point $E(\overline{x},\overline{y},\overline{z})$ to the origin, and for this purpose, make a transformation:

$$\begin{cases} x_1 = x - \overline{x} \\ y_1 = y - \overline{y} \\ z_1 = z - \overline{z} \end{cases}$$

Then, the system (1) is transformed into the following form:

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) \\ \dot{y}_1 = bx_1 + cy_1 - x_1 z_1 - x_1\overline{z} - \overline{x}z_1 \\ \dot{z}_1 = x_1^2 + 2\overline{x}x_1 - hz_1 \end{cases}$$

Apply control u_1,u_2,u_3 to system variable x_1,y_1,z_1 respectively, then the controlled system is

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) + u_1 \\ \dot{y}_1 = bx_1 + cy_1 - x_1 z_1 - x_1\overline{z} - \overline{x}z_1 + u_2 \\ \dot{z}_1 = x_1^2 + 2\overline{x}x_1 - hz_1 + u_3 \end{cases}$$

It can be selected using the ISS control method:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ -x_1^2 \end{bmatrix}$$

The controlled system can be written as follows:

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) + k_1 x_1 \\ \dot{y}_1 = bx_1 + cy_1 - x_1 z_1 - x_1\overline{z} - \overline{x}z_1 + k_2 y_1 \\ \dot{z}_1 = 2\overline{x}x_1 - hz_1 \end{cases} \tag{5}$$

It is only necessary to prove that under certain conditions, the origin of the system (5) is globally asymptotically stable.

If the parameters of the system (5) satisfy such conditions:

$$\begin{cases} k_1 < a \\ 2\max(b + |\overline{z}|, |2\overline{x}|) < -(c + k_2) - |\overline{x}| - M_{x_1} \end{cases} \tag{6}$$

Then the origin of the system (5) is the global asymptotic stability.

Proof: Consider a subsystem of the system (5)

$$\dot{x}_1 = a(y_1 - x_1) + k_1 x_1$$

View y_1, z_1 as the input and x as the state, and define the function $V(x) = \frac{1}{2}x^2$, thus

$$\dot{V} = (k_1 - a)x_1^2 + ax_1 y_1 .$$

Pick any $0 < \varepsilon_1 < a - k_1$ and set $\chi_1(r) = \dfrac{a}{a - k_1 - \varepsilon_1} r$,

Then

$$|x_1| \geq \chi_1(|y_1|) \Rightarrow \dot{V} \leq -\varepsilon_1 x_1^2 .$$

If we take $\underline{\alpha}(r) = \overline{\alpha}(r) = \dfrac{1}{2} r^2$, thus the subsystem (I) is ISS and has a gain function $\gamma_1(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi_1(r) = \dfrac{a}{a - k_1 - \varepsilon_1} r$.

Consider subsystem (II)

$$\begin{cases} \dot{y}_1 = bx_1 + cy_1 - x_1 z_1 - x_1 \overline{z} - \overline{x} z_1 + k_2 y_1 \\ \dot{z}_1 = 2\overline{x} x_1 - hz_1 \end{cases}$$

Define $V(y_1, z_1) = \dfrac{1}{2}(y_1^2 + z_1^2)$, then there is

$$\begin{aligned} \dot{V} &= y_1 \cdot \dot{y}_1 + z_1 \cdot \dot{z}_1 = x_1(by_1 - y_1 \overline{z} + 2\overline{x} z_1) + (c + k_2)y_1^2 - hz_1^2 - \overline{x} y_1 z_1 - x_1 y_1 z_1 \\ &\leq |x_1|(b|y_1| + |\overline{z}||y_1| + |2\overline{x}||z_1|) + (c + k_2)(y_1^2 + z_1^2) + |\overline{x}|(y_1^2 + z_1^2) + M_{x_1}(y_1^2 + z_1^2) \\ &\leq |x_1| \max(b + |\overline{z}|, |2\overline{x}|)(|y_1| + |z_1|) + [|\overline{x}| + M_{x_1} + (c + k_2)](y_1^2 + z_1^2) \\ &\leq 2|x_1| \max(b + |\overline{z}|, |2\overline{x}|)(y_1^2 + z_1^2)^{1/2} + [|\overline{x}| + M_{x_1} + (c + k_2)](y_1^2 + z_1^2) \end{aligned}$$

Pick a sufficiently small ε_2,

When $2\max(b + |\overline{z}|, |2\overline{x}|) < -(c + k_2) - |\overline{x}| - M_{x_1} - \varepsilon_2$, let $\alpha(r) = \varepsilon_2 r^2$ be a K-class function, $\dot{V} \leq -\varepsilon_2(y_1^2 + z_1^2) = -\alpha(\|(y_1, z_1)^T\|)$.

By definition, the function $V(y_1, z_1)$ is the ISS-Lyapunov function of the subsystem (II.), which is stable by theorem 3, the input state of the subsystem (II.). From comment 1, take the gain function as

$$\gamma_2(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi_2(r) = \chi_2(r)$$

Where $\chi_2(r) = \dfrac{2\max(b + |\overline{z}|, |2\overline{x}|)}{-(c + k_2) - |\overline{x}| - M_{x_1} - \varepsilon_2} r$,

Apparently, at this point, $\gamma_2(\gamma_1(r)) = \dfrac{2\max(b + |\overline{z}|, |2\overline{x}|)}{-(c + k_2) - |\overline{x}| - M_{x_1} - \varepsilon_2} \cdot \dfrac{a}{a - k_1 - \varepsilon_1} r < r$ $x(t), y(t), z(t)$ holds.

Therefore, by small gain theorem the system (5) is asymptotically stable at the origin [12].

6. Result of simulation

Taking the initial state of the system as x(0)=y(0)=z(0)=20, we can see from figure 2 thatis bounded and satisfies

$$|x| < 30, |y| < 40, |z| < 80$$

Select $M_{x_1} = 30$ $E_2(\sqrt{h(b+c)}, \sqrt{h(b+c)}, b+c)$. For the equilibrium point, it is noted that at this time a=20, b=14, c=10.6, h=2.8, take the parameter $k_1 = 2, k_2 = -200$ to satisfy (6), and the initial value of the selected system is x(0)=10, y(0)=30, z(0)=-20, and the following figure can be obtained.

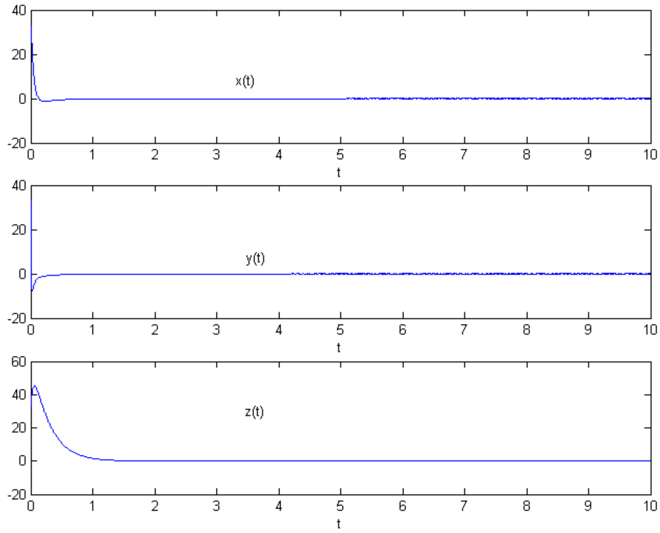

Figure 2: The curve trajectory of $x(t), y(t), z(t)$

7. Conclusion

Through the small gain theorem and ISS control method, the origin of the new chaotic system can be effectively made globally asymptotically stable. Simulation results further confirm this and show that the proposed control strategy is effective.

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