DOI: 10.23977/trance.2024.060416
ISSN 2523-5818 Vol. 6 Num. 4

An Instructional Design for Math Problem Lessons Based on the Alignment of the Problems with Curriculum Standards—"The Cosine Formula for the Sum and Difference of Two Angles" as an Example

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Keywords: Consistency, Instructional Design, Teaching Exercises, High School Math

Abstract: Exercises are an important part of teaching materials, which is the consolidation and deepening of classroom teaching content and provides a platform for students' core literacy in mathematics [1]. The cosine formula of sum and difference of two angles is one of the important contents of trigonometric function in high school mathematics teaching. According to the requirements of the course, students need and can flexibly use these formulas for evaluation and deformation application. The cosine formula of sum and difference of two angles is the basis of trigonometric identical deformation. Other trigonometric function formulas are derived from these basic formulas. Therefore, the exercise should cover all aspects from formula derivation to concrete application, including but not limited to: deriving cosine formula of sum and difference of two angles (such as $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$). Students are able to solve practical problems by applying these formulas, such as using vector or geometric methods for proof. This paper aims to provide a systematic teaching plan for high school mathematics teachers through an in-depth analysis of the teaching content, learning situation, and selection and organization of the "cosine formula of the sum and difference of two angles" [3-7].

1. Presentation of the Issue

In today's educational environment, the exercise class is an important form of teaching and learning, which is given the dual mission of cultivating students' active inquiry ability and improving classroom efficiency. Especially in the field of high school mathematics, the cosine formula of the sum and difference of two angles as a key point of comprehensive application of knowledge, the teaching design of the exercise class is particularly important. [8-10]However, in reality, there is a problem of unclear positioning of the exercises, which is sometimes mixed up with the lectures, and even leads to the students being reduced to problem solving machines and being trapped in the endless sea of problems. The purpose of this paper is to clarify the role and function of the exercise class through the teaching design of the cosine formula of the sum and difference of two angles of the high school mathematics textbook of Xiang Jiao edition, in order to achieve the goal of improving the students' thinking quality, enhancing their problem solving ability, and stimulating their learning initiative, so as to effectively improve the quality of teaching and learning.[11]

2. Content Analysis of Teaching and Learning

The cosine formula for the sum and difference of two angles follows on from the concepts and properties of angles and radians and trigonometric functions.[2] Students have already learned about plane vectors and their applications, and the vector tool facilitates the discovery and proof of the cosine formula of the sum and difference of two angles. The textbook takes the cosine formula of the difference between the two angles as the starting point, and it guides students to explore, and then on this basis, they can obtain other formulas for the sum and difference of the two angles by substitution. Teachers teach according to a structure as shown in Figure 1 to make it more acceptable to students.

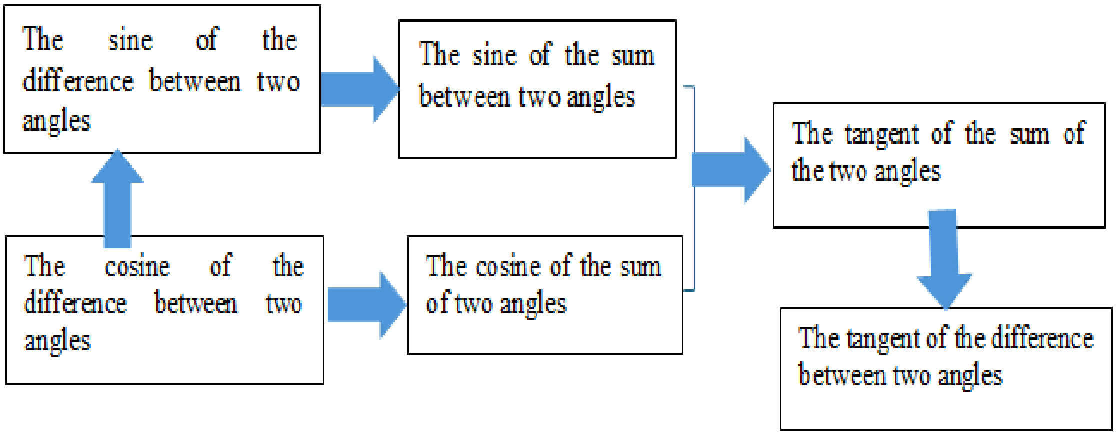

Figure 1: Teaching Structure of Trigonometric Function of Sum and Difference of Two Angles

This logical structure system makes the derivation of other formulas a purely algebraic process, reducing the difficulty of thinking for students. Although it reduces the training of students' creative thinking, it can be compensated by encouraging students to explore independently. The principle of "resolutely removing cumbersome computational processes, artificially tricky problems and overemphasis on minutiae" has been followed.[12]

The 2020 Revision of the General High School Mathematics Curriculum Standards points out that it is necessary to explore and study some identity relationships between trigonometric functions, go through the process of deriving the two-angle difference cosine formula, and know the significance of the two-angle difference cosine formula.[1]

The content arrangement of the knowledge in the textbook is divided into 5 main parts. Part 1 is the introduction of the cosine of the sum and difference of two angles. The textbook begins by suggesting how to make the cosine from an angle that α, β.The trigonometric values of the angles to be solved for the$\alpha - \beta$. The value of the trigonometric function of $\cos(\alpha - \beta)$and $\sin(\alpha - \beta)$? Students arrive at that $\cos(\alpha - \beta) = cos\alpha - cos\beta$ is a common error, in teaching should pay attention to guidance, so that students further clarify the meaning of the "constant formula", but also for this formula as the basis for the derivation of other formulas to prepare. The second part is the derivation of the cosine of the difference between two angles. Teachers will naturally explore the cosine formula of the sum and difference of two angles in connection with the knowledge they have learned, so they can guide students to discover such as constructing congruent triangles in unit circles. Students learn to use the coordinate method to explore the cosine formula of the two-angle difference, but the teacher needs to explain the problems of these methods, and then lead to the cosine formula of using the number product of the vector to explore the two-angle difference.[7]Part 3 explores the cosine formula for the sum of two angles. Students are guided to use the cosine formula for the difference of two angles to derive the cosine formula for the sum of two angles. Part 4 is to identify the cosine formula for the sum and difference of two angles. By observing the structure of the cosine formula of the sum of the two angles and the cosine formula of the difference between the two angles, the teacher should guide the students to carry out a discernment analysis in order to better grasp the students, and the teacher should let the students master the consistency of the formula in structure and the difference of symbols, so as to be able to use the formula accurately. The fifth part is the teaching of examples. Examples of the formula is mainly a forward application and reverse application.[13-15]

3. Analysis of the Learning Situation

Students have mastered and can apply the cosine formula for the sum and difference of two angles, simplifying complex expressions, and solving certain real-world problems prior to working on the exercises. Students also have basic knowledge of the fundamental concepts of trigonometry and basic constants. However, students at this stage still have the following problems.

1) Memory impairment: Students do not have a firm memory of formulas, especially the symbols.

2) Application difficulties: Students have difficulty identifying when and how to apply formulas, and lack the ability to effectively transform conditions.

3) Calculation errors: Students often make mistakes due to carelessness in numerical calculations.

4) Comprehensive ability: When faced with a multi-knowledge comprehensive question type, students' solution strategies are insufficient.

4. Selection and Organization of Topics

It is pointed out in the standard that exercises are an important part of the teaching materials, and it is necessary to improve the effectiveness of exercises, scientifically and accurately grasp the difficulty of the exercises, and develop questions with applicability, openness, and exploratory nature. Exercise class is the consolidation and deepening of classroom teaching content, and should also provide a platform for students to develop core literacy in mathematics.[1]For example, the practice problems of the exercise class should pay attention to the hierarchy of the problems, from shallow to deep, to help students master the knowledge and skills at the same time, to further understand the basic ideas of mathematics, and to accumulate the experience of mathematical thinking; Teachers should pay attention to the creation of situations and problems, which is conducive to the understanding of the nature of mathematical knowledge, and to enhance the core qualities of the discipline of mathematics, and so on.[6]Based on the requirements of the standard and the analysis of the teaching content, it is found that the sample exercises in the Xiang jiao edition of the high school mathematics textbook are highly consistent with the standard. Therefore, we choose the following exercises in the textbook.

1) Find the cosine of the angle 75°, 15°.

2) Find the value of the following equation.

(a)$cos20°cos25° - sin20°sin25°$

(b)$\sin5°\cos40° + \sin85°\sin40°$

(c)$\cos^2 22.5° - \sin^2 22.5°$

3) Known $sin\alpha = \frac{4}{5}$ $, cos\beta = \frac{5}{13}$, and the angle α, β are located in the second and fourth quadrants, respectively, and find the value of the $\cos(\alpha + \beta)$ and $\cos(\alpha - \beta)$.

4) In the ΔABC, it is known that $sinA = \frac{3}{5}$,$cosB = \frac{5}{13}$, try to find out the value of the $cosC$.

5) Using the cosine formula for the difference between two angles, prove the following induction formula.

(a)$\sin\alpha = \cos(\frac{\pi}{2} - \alpha)$

(b)$\cos\alpha = \sin(\frac{\pi}{2} - \alpha)$

Through the investigation of students, we found that the students for the exercise class demand is not only to explain the exercises in the book, but also students hope that the teacher can choose some targeted and flexible, can cover more knowledge points of the exercises. At the same time, teachers can pay attention to the difficulty of the exercises gradient design, to meet the needs of different students. Therefore, we have selected two comprehensive questions related to the knowledge points of this section outside the textbook.

1) It is known that the vertex of the angular α coincides with the origin O, the start edge coincides with the non-negative semi-axis of the x-axis, and the terminal edge passes through the point $P(\frac{3}{5}, \frac{4}{5})$.

(a) Solve for the value of the $\sin(\alpha + \pi)$ and $cos2\alpha$.

(b) If the angle β satisfies $cos\beta = -\frac{5}{13}$ and the angle β is the third quadrant angle, find the value of $\cos(\alpha + \beta)$.

2) In Δ ABC, a, b,and c are the opposite sides of angles A, B, and C, respectively, $b = \sqrt{7}, c = 2, B = \frac{\pi}{3}$

(a) Solve for the value of the a.

(b) Solve for the value of the $sinA$

(c) Seek for the value of the $\cos(B - A)$.

5. Teaching Objectives and Key Points

Teaching Objective.

Students can use the cosine formula of the sum and difference of two angles, and can apply them to simplify and find the value of trigonometric functions. Students stimulate their interest in learning through real-life examples, feel mathematical ideas, and improve their logical reasoning, mathematical abstraction, and mathematical abstract literacy.

Key Points: Inverse application of the cosine formula for the sum and difference of two angles.

6. The Teaching and Learning Process

(1) Reviewing old knowledge

Teachers' activities: Teachers guide students to recall key knowledge, check the learning quality of students' new lectures, and receive timely feedback so that they can adjust the teaching strategy and time arrangement of exercise classes at any time.

Student Activity: Students review the knowledge points related to the cosine formula of the two angles and the difference to deepen their memory and understanding of the formula.

$$\cos(\alpha + \beta) = cosa\ cos\beta - sin\alpha\ sin\beta \qquad (C_{(\alpha+\beta)}) \qquad (1)$$

$$\cos(\alpha - \beta) = cosa\ cos\beta + sin\alpha\ sin\beta \qquad (C_{(\alpha-\beta)}) \qquad (2)$$

The cosine theorem

$$a^2 = b^2 + c^2 - 2bc\, cosA \quad (3)$$

$$b^2 = a^2 + c^2 - 2ac\, cosB \quad (4)$$

$$c^2 = a^2 + b^2 - 2ab\, cosC \quad (5)$$

The sine theorem

$$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} \quad (6)$$

Design intention: Students clarify their teaching ideas, and the teacher leads students to review their old knowledge and clarify the knowledge points required for the problems solved in this exercise.

(2) Dealing with textbook examples and exercises

1) Find the cosine of the angle 75 °, 15 °

2) Find the value of the following equation.

(a)$\cos 20°\cos 25° - \sin 20°\sin 25°$

(b)$\sin 5°\cos 40° + \sin 85°\sin 40°$

(c)$\cos^2 22.5° - \sin^2 22.5°$

Teacher Activity: The teacher asks the students to solve the problem based on the knowledge they have just reviewed.

Student Activity: Students think and do calculations.

Design intent: Question 1 is a direct application of the formula, in the absence of special instructions, the value of the trigonometric function are reduced to the value of the trigonometric function of the special angle, so that students in the process of doing the problem to form this consensus. The purpose of question 1 is to familiarize students with the specific application of the cosine formula for the sum and difference of two angles.

Question 2 is a reverse application of the formula. (1) equation fits exactly the "right" side of the cosine formula of the sum of the two angles, and the result can be obtained by applying the formula in reverse. Equation (2) does not fully fit the cosine formula of the sum and difference of the two angles, and needs to be transformed to a certain extent. The teacher can let the students explore how to form the formula (1) independently, and guide the students to review the induction formula $cos\alpha = \sin(\frac{\pi}{2} - \alpha)$, and then the students can get $sin5° = cos85°$ from the induction formula, so that the students can have the solution to the problem (2).

3) Known $sin\alpha = \frac{4}{5}$ $, cos\beta = \frac{5}{13}$, and the angle $\alpha, \beta$are located in the second and fourth quadrants, respectively, and find the value of the $\cos(\alpha + \beta)$and$\cos(\alpha - \beta)$.

4) In ΔABC, it is known that $sinA = \frac{3}{5}$,$cosB = \frac{5}{13}$, try to find out the value of the $cosC$.

Teacher's activity: The teacher leads the students to do the third question, sort out the ideas for solving the problem, and detail the process of solving the problem. After completing the exercise, the teacher asked how to analyze the problem if there is no "corner α, β are in the second and fourth quadrants".

Student Activity: Students think about the teacher's problem and follow the teacher's ideas to understand the practice of this type of question and solve problem 4.

Design intent: These two questions are also a direct application of the formula, but more than the previous exercises using the same angle trigonometric relationship to find the trigonometric value of the process. Through these two questions, students will cultivate the idea of classification and discussion, deepen their understanding and mastery of formulas, and cultivate the rigor of thinking.

5) Using the cosine formula for the difference between two angles, prove the following induction

formula.

(a)$sin\alpha = \cos(\frac{\pi}{2} - \alpha)$

(b)$cos\alpha = \sin(\frac{\pi}{2} - \alpha)$

Teacher activities: Question 5 is a simple proof question in the textbook, after the implementation of the above teaching links, the difficulty of the exercise problem has not constituted a huge obstacle. The teacher needs to explain to the students the method of proving the equation, the left side through a certain change to get the right formula, or from the right side to make a certain change to get the left formula. The teacher leads the students to do (1).

Student Activity: Students do what the teacher thinks (2).

Design intention: Students have already learned the induction formula, through the cosine formula of the difference between two angles, can let students better understand what they have learned before.

6) (From the midterm exam paper of the first semester of the senior high school of the 35th middle school in Beijing in 23) It is known that the vertex of the angular α coincides with the origin O, the start edge coincides with the non-negative semi-axis of the x-axis, and the terminal edge passes through the point $P(\frac{3}{5}, \frac{4}{5})$.

(a) Solve for the value of the $\sin(\alpha + \pi)$ and $cos2\alpha$.

(b) If the angle β satisfies $cos\beta = -\frac{5}{13}$ and the angle β is the third quadrant angle, the value of $\cos(\alpha + \beta)$ is obtained.

7) (from the joint examination paper for the mid-term of the first semester of the senior secondary school of the key schools in Tianjin) 2) In Δ ABC, a, b,and c are the opposite sides of angles A, B, and C, respectively, $b = \sqrt{7}, c = 2, B = \frac{\pi}{3}$

(a) Solve for the value of the a.

(b) Solve for the value of the $sinA$.

(c) Seek for the value of the $\cos(B - A)$.

Teaching Activities: Based on the previous exercises, students have mastered the application of the cosine formula of the sum and difference of two angles. When reviewing and consolidating in class, the teacher has already led the students to review not only the cosine formula of the sum and difference of two angles, but also what they have learned in the past. For question 6.7, students are given enough time to clear their own ideas before being led through the board. The teacher then leads the students to explain the board book.

Student Activity: Question 6.7 is a comprehensive question, covering the previously learned induction formula, sine and cosine theorem. Students solve problems independently and discover their weaknesses throughout the problem-solving process. After solving the exercises, students are follow the teacher's ideas to check and fill in the gaps.

Design intention: Through understanding the students' demand for exercises, students prefer to do some comprehensive questions. Therefore, in addition to the textbook, we have selected a more suitable type of problems for students. The whole set of exercises are from simple to difficult, with a hierarchy. It helps students to further understand mathematical ideas and accumulate experience in mathematical thinking while mastering knowledge and skills. Through a series of mathematical problems, it is conducive to students' understanding of the nature of mathematics and enhancement of the core qualities of mathematics. The selected exercises are also concerned with the connection between the main lines of mathematical content, which is conducive to students' organizing their understanding and systematic mastery of the mathematical content they have learned.

7. Reflection on Teaching and Learning

In teaching, the teacher further consolidates and deepens the students' understanding and application ability of formulas by guiding them to practice exercises. There are two main problems in the exercise course: in the selection of exercises, it is necessary to pay attention to comprehensive coverage and clear hierarchy. Through different types of difficult and type questions, the teacher tries to guide students to understand and apply the cosine formula of the sum and difference of two angles from multiple perspectives. However, in the process of teaching, some questions are too simple, and students with a good foundation are not attentive, and their thoughts will "break down". Therefore, teachers should pay more attention to the actual situation of students when choosing exercises to ensure that the questions are of moderate difficulty, and these exercises can stimulate students' interest in thinking, and at the same time not make them feel frustrated and take care of the overall situation. In the process of explaining the exercises, a variety of teaching methods are used: students think independently, group discussions and teacher explanations are combined. This approach is designed to foster students' ability to learn independently and cooperatively. However, in practice, student engagement was uneven, with some students being active and others being passive. In the future, more attention should be paid to stimulating students' interest and enthusiasm in learning, and students should be encouraged to actively participate in the explanation and discussion of exercises through a variety of teaching methods.

8. Conclusions

The cosine formula of sum and difference of two angles is a basic knowledge point in high school mathematics. Its derivation and application are of great significance to students' understanding and trigonometric function Reasonable exercise design can effectively improve students' problem-solving ability and ensure the consistency between exercises and courses This will not only help students get good grades in exams, but also lay a solid foundation for their future study and life.

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