Transactions on Computational and Applied Mathematics (2024)
Clausius Scientific Press, Canada
DOI: 10.23977/tracam.2024.040115
ISSN 2616-1826 Vol. 4 Num. 1

The Decomposition of Pentagonal Numbers

Fuju Wu[1], Deyi Chen[2]

[1]*Wuhan Britain-China School, Wuhan, Hubei, 430030, China*
[2]*Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang, 310027, China*

Keywords: Pentagonal Numbers; Elliptic curves; Pell's equation

Abstract: In this paper, we have demonstrated that there are infinitely many pentagonal numbers which have two different ways to be decomposed as the product of two non-1 pentagonal numbers, with the domain being in positive rational numbers. This was achieved by transforming an equation into an elliptic curve, identifying a rational point on this curve, and subsequently employing the Nagell-Lutz Theorem to establish the existence of infinitely many rational points on the elliptic curve. Finally, we conjecture that if the domain is restricted to positive integers, then there do not exist such two different decompositions.

1. Introduction

People have always been interested in certain special shaped numbers, such as Triangle numbers

$$T(n)=\frac{n(n+1)}{2}, \tag{1}$$

Pentagonal numbers

$$P(n)=\frac{n(3n-1)}{2}, \tag{2}$$

and k-gonal numbers

$$P_k(n)=(k-2)\frac{n(n-1)}{2}+n \tag{3}$$

Where $n\in\mathbb{Z}_+$.

In 1783, Euler [1] proved a theorem related to the partition of integers and pentagonal numbers:

$$1+p(1)q+p(2)q^2+\cdots=\frac{1}{\sum_{n=-\infty}^{+\infty}(-1)^n q^{\frac{n(3n-1)}{2}}} \tag{4}$$

where $p(n)$ represents the number of partitions of n. In 1831, Cauchy [2] proved that any positive integer can be represented as the sum of up to n n-gonal numbers. Deyi Chen and Tianxin Cai [3] proved that there are infinitely many triangular numbers which have two different ways to be decomposed as the product of two triangular numbers, each greater than 1. For example,

$$\binom{36}{2}=\binom{3}{2}\binom{21}{2}=\binom{4}{2}\binom{15}{2},$$
$$\binom{1225}{2}=\binom{15}{2}\binom{120}{2}=\binom{21}{2}\binom{85}{2}. \tag{5}$$

A natural question is whether there is a similar result for pentagonal numbers. In this paper, we prove

Theorem 1: There are infinitely many pentagonal numbers that can be decomposed into the product of two non-1 pentagons, with the domain being in positive integers $\mathbb{Z}_+$, i.e. there are infinitely many $(x, y, z) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+, y > 1, z > 1$ such that

$$P(x) = P(y)P(z). \tag{6}$$

Are there infinitely many pentagonal numbers which have two different ways to be decomposed as the product of two non-1 pentagonal number? We performed some numerical calculations but did not find such examples. However, if we extend the domain $(x, y, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ we found

$$\begin{gathered} P(35) = P(-13)P(-2) = P(-4)P(7), \\ P(3267) = P(-18)P(147) = P(15)P(180), \\ P(3780) = P(-135)P(23) = P(12)P(261), \\ P(67947) = P(-16)P(3432) = P(48)P(1160), \\ P(-3553) = P(-11)P(260) = P(-4)P(697), \\ P(-765) = P(-68)P(-9) = P(-45)P(14), \\ P(-693) = P(-117)P(5) = P(12)P(48). \end{gathered} \tag{7}$$

Note that in the first equation $P(35) = P(-13)P(-2) = P(-4)P(7),$ the number $-13, -2, -4, 7,$ satisfies

$$3a - 1, b, a, -3b + 1 \tag{8}$$

where $a = -4, b = -2.$ Hence, we consider

$$P(c) = P(3a - 1)P(b) = P(a)P(-3b + 1). \tag{9}$$

Noting that

$$P(3a - 1)P(b) - P(a)P(-3b + 1) = \frac{(3a-1)(3b-1)(a-2b)}{2}, \tag{10}$$

let $a = 2b$, we have the identity

$$P(6b - 1)P(b) = P(2b)P(-3b + 1). \tag{11}$$

Therefore, we only need to consider

$$P(c) = P(2b)P(-3b + 1). \tag{12}$$

Noting that $1 + 24P(c) = (6c - 1)^2$,

$$1 + 24P(-3b - 1)P(2b) = 1944b^4 - 1404b^3 + 324b^2 - 24b + 1 = (6c - 1)^2, \tag{13}$$

so we consider the elliptic curve

$$y^2 = 1944x^4 - 1404x^3 + 324x^2 - 24x + 1 \tag{14}$$

where $x = b, y = 6c - 1$. By using rational points on the elliptic curve, we prove

Theorem 2: There are infinitely many pentagonal numbers which have two different ways to be decomposed as the product of two non-1 pentagonal numbers, with the domain being in positive rational numbers $\mathbb{Q}_+$, i.e. there are infinitely many $(x, y, z, u, v) \in \mathbb{Q}_+^5$ such that

$$P(x) = P(y)P(z) = P(u)P(v). \tag{15}$$

where $y, z, u, v \neq 1, \{P(y), P(z)\} \neq \{P(u), P(v)\}$.

And we have the following

Conjecture: There is no pentagonal number which has two different ways to be decomposed as the product of two non-1 pentagonal numbers, with the domain being in positive integers $\mathbb{Z}_+$, i.e. there is no $(x, y, z, u, v) \in \mathbb{Z}_+^5$, such that

$$P(x) = P(y)P(z) = P(u)P(v). \tag{16}$$

where $y, z, u, v \neq 1, \{y, z\} \neq \{u, v\}$.

2. Preliminaries.

Lemma 1 [4] If the quartic curve

$$y^2 = ax^4 + bx^3 + cx^2 + dx + e \tag{17}$$

has a rational point, it is birational equivalent to a cubic curve

$$Y^2 = 4X^3 - g_2X - g_3. \tag{18}$$

In particular,

$$y^2 = x^4 - 6cx^2 + 4dx + e \tag{19}$$

is birational equivalent to (18), where $g_2 = e + 3c^2, g_3 = -ce - d^2 + c^3$.

Lemma 2 Elliptic curves

$$E_1(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 \mid y^2 = 1944x^4 - 1404x^3 + 324x^2 - 24x + 1\} \tag{20}$$

and

$$E_2(\mathbb{Q}) = \{(U, V) \in \mathbb{Q}^2 \mid V^2 = U^3 - 9072X + 291600\} \tag{21}$$

are birationally equivalent.

Proof: In $E_1(\mathbb{Q})$, divide both sides by x^4,

$$\left(\frac{y}{x^2}\right)^2 = 1944 - 1404\frac{1}{x} + 324\frac{1}{x^2} - 24\frac{1}{x^3} + \frac{1}{x^4}. \tag{22}$$

Let $X_1 = \frac{1}{x}, Y_1 = \frac{y}{x^2}$, (22) changes to

$$Y_1^2 = X_1^4 - 24X_1^3 + 324X_1^2 - 1404X_1 + 1944. \tag{23}$$

Let $X_1 = X_2 + 6, Y_1 = Y_2$, (23) changes to

$$Y_2^2 = X_2^4 + 108X_2^2 - 756X_2 + 1296. \tag{24}$$

By lemma 1, (24) birational equivalent to

$$Y^2 = 4X^3 - 2268X + 18225. \tag{25}$$

(25) multiplied by 4^2 on both sides, we have

$$(4Y)^2 = (4X)^3 - 9072(4X) + 291600. \tag{26}$$

Let $V = 4Y, U = 4X$, (26) changes to $E_2(\mathbb{Q})$.

Remark: $\left(\frac{81}{4}, \frac{2727}{8}\right) \in E_2(\mathbb{Q})$.

Lemma 3 (Nagell–Lutz Theorem) [5] Let

$$y^2 = x^3 + ax^2 + bx + c \tag{27}$$

be a non-singular cubic curve with integer coefficients a, b, c, and $P(x, y)$ be a rational point of finite order. Then x and y are integers.

3. Proofs of the theorems.

Proof of Theorem 1.

We prove that

$$P(x) = P(2)P(y) \tag{28}$$

has infinitely many positive integer solutions.

Noting $1 + 24P(x) = (6x-1)^2$,

$$(28) \Leftrightarrow \frac{(6x-1)^2-1}{24} = 5\frac{(6y-1)^2-1}{24} \Leftrightarrow (6x-1)^2 - 5(6y-1)^2 = -4. \tag{29}$$

Let $6x - 1 = X, 6y - 1 = Y$, (29) changes to

$$X^2 - 5Y^2 = -4. \tag{30}$$

Some positive integer solutions of (30) are given by

$$X_n + Y_n\sqrt{5} = \left(1 + \sqrt{5}\right)\varepsilon^n \tag{31}$$

where $\varepsilon = 9 + 4\sqrt{5}, n \geq 0.$ Noting $\varepsilon^2 = 18\varepsilon - 1$,

$$(1+\sqrt{5})\varepsilon^{n+2} = 18(1+\sqrt{5})\varepsilon^{n+1} - (1+\sqrt{5})\varepsilon^n, \tag{32}$$

i.e.

$$X_{n+2}+Y_{n+2}\sqrt{5}=18(X_{n+1}+Y_{n+1}\sqrt{5})-(X_n+Y_n\sqrt{5}). \tag{33}$$

The recursive sequences derived from (33) are

$$\begin{cases} X_{n+2} = 18X_{n+1} - X_n, X_0 = 1, X_1 = 29 \\ Y_{n+2} = 18Y_{n+1} - Y_n, Y_0 = 1, Y_1 = 13 \end{cases} \tag{34}$$

But X_n and Y_n modulo 6 form a periodic sequence, i.e.

$$\begin{cases} X_n \bmod 6 = 1,5,5,1,1,5,5,1,\cdots \\ Y_n \bmod 6 = 1,1,5,1,1,1,5,1,\cdots \end{cases} n \geq 0, \tag{35}$$

then

$$X_n \equiv Y_n \equiv 5 \bmod 6 \Leftrightarrow n = 4k-2 \tag{36}$$

where $k \in \mathbb{Z}_+$.
Hence

$$x_{4k-2} = \frac{X_{4k-2}+1}{6} \in \mathbb{Z}_+, y_{4k-2} = \frac{Y_{4k-2}+1}{6} \in \mathbb{Z}_+. \tag{37}$$

and

$$P(x_{4k-2}) = P(2)P(y_{4k-2}) \tag{38}$$

where $k \in \mathbb{Z}_+$. We have completed the proof of Theorem 1
Proof of Theorem 2.
By lemma 3 and $\left(\frac{81}{4}, \frac{2727}{8}\right) \in E_2(\mathbb{Q})$, it follows that $E_2(\mathbb{Q})$ has infinitely many rational points.
By lemma 1, $E_1(\mathbb{Q})$ has infinitely many rational points (x, y).
By introduction

$$P(c) = P(6b-1)P(b) = P(2b)P(-3b+1) \Leftrightarrow y^2 = 1944x^4 - 1404x^3 + 324x^2 - 24x + 1 \tag{39}$$

where $x = b, y = 6c-1$. Hence

$$P(c) = P(6b-1)P(b) = P(2b)P(-3b+1) \tag{40}$$

has infinitely many rational solutions where $b = x, c = \frac{y+1}{6}$.
According to the fact that

$$\begin{cases} P(6b-1) = P(2b) \\ P(b) = P(-3b+1) \end{cases} \Leftrightarrow b = \tfrac{1}{4} \text{ and } \begin{cases} P(6b-1) = P(-3b+1) \\ P(b) = P(2b) \end{cases} \Leftrightarrow b = \tfrac{1}{9} \tag{41}$$

There are infinitely many b such that $\{P(6b-1), P(b)\} \neq \{P(2b), P(-3b+1)\}$. Finally, since $P(r) = P\left(\frac{1}{3} - r\right)$, we can always obtain positive rational solutions. We have completed the proof of Theorem 2.

References

[1] Euler, On the remarkable properties of the pentagonal numbers. Acta Academia of the Science of the Empire of Petropolitan, 1783, 4(1):56-75.

[2] Heath, Sir Thomas Little, Diophantus of Alexandria: A study in the history of Greek algebra, Cambridge University Press, 1910:188.

[3] Chen D, Cai T. The decomposition of triangular numbers [J]. The Rocky Mountain Journal of Mathematics, 2013: 75-81.

[4] Mordell L J. Diophantine equations: Diophantine Equations [M]. Academic press, 1969.

[5] Silverman J H, Tate J T. Rational points on elliptic curves [M]. New York: Springer-Verlag, 1992.