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Walsh Functions with Barycentric Symmetry over Triangular Domain

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Abstract: The Walsh function is a complete set of orthogonal functions defined on the interval [0, 1], with values fixed at either +1 or -1. The function demonstrates the beauty of symmetry in mathematics and plays a unique role in applications. This paper focuses on the Walsh function over the trigonometric domain and develops a set of completely orthogonal functions over the trigonometric domain. The results illustrate the symmetry of these functions in both mathematical form and the barycentric coordinate system.

1. Introduction

The Walsh function is a complete orthogonal system of functions on the interval [0,1] formulated by the American mathematician J.L. Walsh. Each function only takes the value 1 or -1, exhibiting a square wave shape. Unlike functions that pursue continuous differentiability in traditional calculus, Walsh functions were initially constructed only as mathematical “counterexamples”. However, in the 1970s, with the widespread application of semiconductor technology and large-scale integrated circuits, Walsh functions [1-2] laid the foundation for practical applications and demonstrated unique efficacy, sparking a research boom on Walsh functions in the field of signal processing. In order to extend the Walsh function to the multivariable cases, a direct way is to define it in the form of a tensor product. Although the generalization process is not complicated, it is still essentially a one-dimensional simple inference. The purpose of this paper is to extend the Walsh function to the trigonometric domain, that is, to study the construction of the Walsh function over the trigonometric domain. The Walsh function over the triangular domain was first seen in [3]. In [4], Qi constructed another p-order Walsh function over the trigonometric domain. Specifically, by uniformly dividing the edges of the triangular domain, several sub-triangles were formed, and then each row of symbolic vectors of the Hadamard matrix was filled into the sub triangles respectively according to the prescribed ordering principle, thus obtaining the Walsh function in H-order over the triangular domain.

However, after the H-order Walsh function is extended to a triangular domain, the Walsh function loses its original symmetry, which is a drawback. A function system with good symmetry not only makes the function structure clearer but also plays an important role in fast algorithm design and other applications.

2. One-dimensional and Two-dimensional Walsh Functions

Walsh functions are a class of complete orthogonal function systems on the interval [0,1]. Initially, Walsh defined the Walsh function by "compressing and forward/reverse copying" in his paper. Common Walsh functions mainly include Walsh order, P-order, and H-order. Among them, Walsh order is defined using the generation method of Gray code and Rademacher function. The generated Walsh function has the characteristic of an increasing number of sign changes. The first eight Walsh functions are shown in Figure 1.

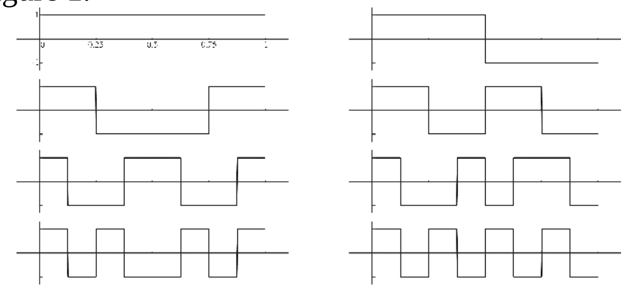

Figure 1: The first eight Walsh functions.

The Walsh function on the plane region$[0,1]^2$ is now defined as

$$\varphi_{i,j}(x,y) = wal(i,x)wal(j,y)$$

$$i = 0,1,2,\cdots.2^m - 1, \qquad j = 0,1,2,\cdots,2^n - 1$$

It is called a tensor product Walsh function of two variables.

3. Walsh Function of Barycentric Symmetry over Trigonometric Domain

The method of generating Walsh functions over the triangular domain is to combine the characteristics of generating P-order or H-order Walsh functions in one-dimensional cases with the area coordinates under a self-similar subdivision of the triangular domain. The Walsh functions of P-order and H-order over the triangular domain obtained by this method also have orthogonality and completeness.

One-dimensional Walsh functions have good symmetry, that is, any basis function is always odd or even symmetric about its center point. Therefore, in this article, a mutually symmetric Walsh function over a triangular domain will be defined.

The self-similar subdivision of the triangle domain can be divided by connecting the midpoints of each side of the triangle, and four congruent subtriangles similar to the original triangle are obtained, as shown in Figure 2, which are denoted as

$$\triangle=\triangle_1^0\cup\triangle_1^1\cup\triangle_1^2\cup\triangle_1^3$$

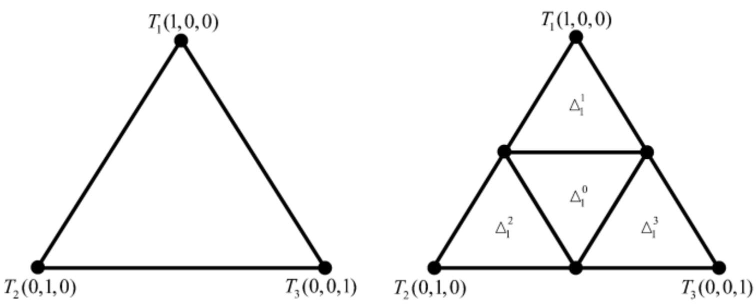

Figure 2: Self-similar subdivision of the triangle domain.

The commutation set over the trigonometric field is recorded as $\{\sigma_1, \sigma_2, \sigma_3\}$, i.e

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Then $\{f_1(\eta_1, \eta_2, \eta_3), f_2(\eta_2, \eta_3, \eta_1), f_3(\eta_3, \eta_1, \eta_2)\}$ is called a set of barycentric symmetric functions over the triangular domain $\triangle$.

Δ is recorded as a triangular domain with an area equal to 1, and the Walsh function system includes a function defined on Δ with a value of 1. $W_{n,j}^{k}(\eta)$ is recorded as the j Walsh function generated by $W_{0,0}^{k}(\eta)$ in layer n. When n = 0, the Walsh function is defined as

$$W_{0,0}^{1}(\eta) = \begin{cases} 1, & \eta \in \triangle_1^0 \cup \triangle_1^1 \\ -1, & \eta \in \triangle_1^2 \cup \triangle_1^3 \end{cases} \tag{1}$$

According to the barycentric symmetry, $W_{0,0}^{2}(\eta)$ and $W_{0,0}^{3}(\eta)$ can be obtained as:

$$W_{0,0}^{2}(\eta) = \begin{cases} 1, & \eta \in \triangle_1^0 \cup \triangle_1^2 \\ -1, & \eta \in \triangle_1^1 \cup \triangle_1^3 \end{cases} \tag{2}$$

$$W_{0,0}^{3}(\eta) = \begin{cases} 1, & \eta \in \triangle_1^0 \cup \triangle_1^3 \\ -1, & \eta \in \triangle_1^1 \cup \triangle_1^2 \end{cases} \tag{3}$$

Consequently, the 4th order Walsh matrix is obtained as:

$$\mathcal{W} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{4}$$

It is easy to verify that $\{W_{0,0}^{1}(\eta), W_{0,0}^{2}(\eta), W_{0,0}^{3}(\eta)\}$ are orthogonal to each other, that is, when $k_1 \neq k_2$

$$\int_{\triangle} W_{0,0}^{k_1}(\eta) W_{0,0}^{k_2}(\eta)\, d\eta = 0, \qquad k_1, k_2 \in \{1,2,3\}$$

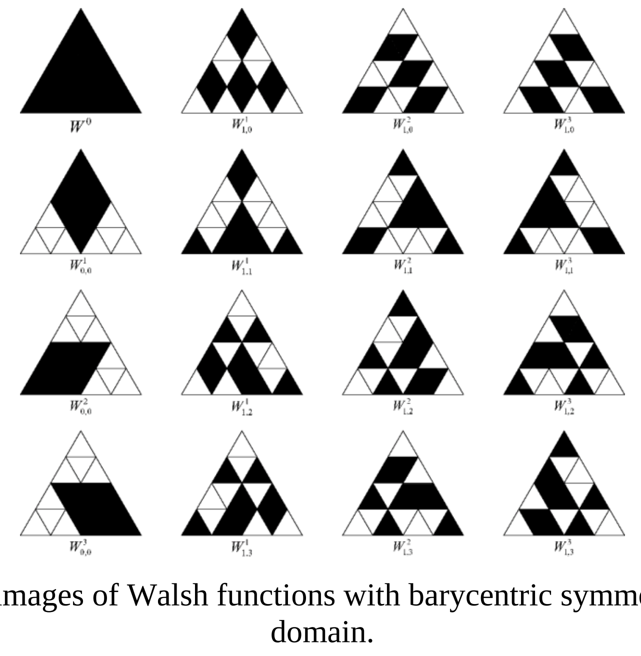

Figure 3: The first 16 images of Walsh functions with barycentric symmetry over the triangular domain.

Starting from layer n=1, the Walsh functions in each layer are compressed to $\triangle_1^0$,$\triangle_1^1$,$\triangle_1^2$, and $\triangle_1^3$ respectively by each Walsh function in the previous layer, and the components of each row vector in the 4-order Walsh matrix are assigned to the four compressed functions. Therefore, each Walsh function in the $n-1$ layer could generate four Walsh functions in layer n. Among them, the lth (l = 0, 1, 2, 3) function is:

$$W_{n,4j+l}^k(\eta) = \mathcal{W}_{l_i} W_{n-1}^k(M_i^{-1}\eta), \qquad \eta \in \triangle_1^i$$

As mentioned above, k can be set to 1 according to the properties of barycentric symmetry, and for k = 2, 3, it can be generated by barycentric symmetry in the same way as Equations (2) and (3). Figure 3 shows the first 16 images of Walsh functions with barycentric symmetry over the triangular domain. It is evident that each group of $\{W_{n,j}^1(\eta), W_{n,j}^2(\eta), W_{n,j}^3(\eta)\}$ is barycentric symmetric on the triangular domain.

4. Experiments and Analysis

For the general image composed of square pixels, the Walsh function of the tensor product can get better results, but for the image originally composed of triangle pixels, the Walsh function of the tensor product has poor processing results. Better results can be obtained by using the Walsh function of barycentric symmetry over the triangular domain constructed in this paper.

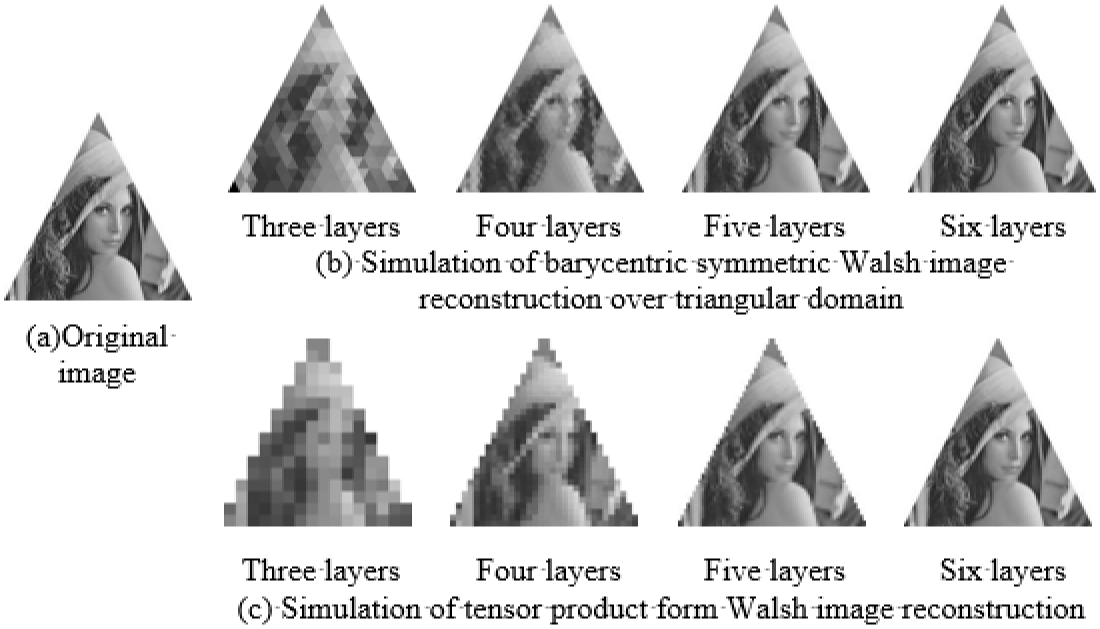

Figure 4: Using the first N layers of Walsh functions to reconstruct the Lena image simulation results.

In this work, using the tensor product-based Walsh function and barycentric symmetry-based Walsh function, the reconstruction effects of the two methods under the condition of 100% sampling rate were compared. In addition, the reconstruction effects of the two methods were also investigated under the condition of using different numbers of basis functions. The reconstruction effects were compared in terms of visual effects and peak signal-to-noise ratio (PSNR).

Under the condition of a 100% sampling rate, an image containing 65,536 triangular pixels was used as the original image, and the Walsh function with 3, 4, 5, and 6 layers was used respectively to compare the reconstruction effects of different methods. The simulation effects of reconstruction are shown in Figure 4.

Table 1: Peak signal-to-noise ratio (PSNR) of image reconstruction at 100% sampling rate.

	3 layers	4 layers	5 layers	6 layers
the Walsh function with barycentric symmetry	18.878	21.231	23.776	27.696
the Walsh function in tensor product form	18.172	20.149	22.645	25.811

The Table 1 show that under the condition of a 100% sampling rate, the reconstruction effect of triangle pixel images using the Walsh function with barycentric symmetry over the triangle domain is better than using the Walsh function in tensor product form with the same number of basis functions.

5. Conclusions

The Walsh function constructed in this article has the property of barycentric symmetry, which is not only aesthetic in mathematical form but also makes some progress in reconstructing images over the triangular domain. Future work can explore a wider range of function categories and their properties in barycentric symmetry. These studies are expected to deepen the understanding of functions in the triangular domain and provide more abundant mathematical tools for application fields such as signal processing and image processing.

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