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Illustrated Difficulties in Teaching Linear Algebra

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Abstract: This study aims to explore the application and effectiveness of a geometric perspective in teaching linear algebra. Traditional linear algebra education often emphasizes algebraic calculations and symbolic derivations, while neglecting the important geometric applications of linear algebra. It is crucial to integrate the idea of connecting numbers and shapes in linear algebra education, combining concepts such as matrices, vectors, with geometric concepts such as planes, lines, and spaces, to help students better understand and apply their knowledge of linear algebra. Using a geometric perspective in teaching linear algebra can enhance students' learning interest and effectiveness, strengthen their spatial imagination and creativity, and assist them in mastering and applying their knowledge of linear algebra. This research has certain guiding significance and practical value for improving the teaching of linear algebra courses.

1. Introduction

Linear algebra is primarily the study of finite-dimensional vector spaces and their algebraic structures of linear transformations. In many years of engineering linear algebra education, students commonly express that linear algebra is a subject filled with concepts and theorems, abstract and difficult to understand, and knowledge that is challenging to grasp. They are unclear about the practical applications of linear algebra. The reason for this lies in the abstract nature of linear algebra, where many conclusions cannot be concretely visualized. It becomes particularly important to handle the relationship between linear algebra knowledge and geometric images.

In order to establish a teaching philosophy of 'geometry without algebra', and to help students develop a concrete and visual understanding of linear algebra, we explore a teaching model that combines spatial geometry with linear algebra. This ensures that educational ideas, content, methods, means, and models adapt to the progress of the times, enabling students to benefit from lifelong learning in the field of linear algebra.

In the following, we will combine the more abstract content of linear algebra education with the integration of numbers and shapes, helping students transform abstract mathematical concepts into tangible spatial geometric understanding. The guidance of teachers is indispensable in this process.

2. Linear dependence and independence

In teaching, the essence of a definition may not always be clear, such as the concepts of linear dependence and independence in linear algebra. Linear spaces are based on the concepts of linear dependence and independence. If these concepts are not understood at their core, it will undoubtedly affect students' understanding of linear space knowledge. Therefore, when teaching the relationships between vectors, it is necessary to focus on explaining the spatial geometric relationships between vectors.

Definition 1: Let there be a set of n-dimensional vectors $\alpha_1,\alpha_2,\ldots\alpha_m$. If there exists a non-trivial solution where $k_1,k_2,\ldots k_m$ are not all zero, such that $k_1\alpha_1+k_2\alpha_2\ldots+k_m\alpha_m=0$, then the vector set $\alpha_1,\alpha_2,\ldots\alpha_m$ is called linearly dependent; otherwise, it is called linearly independent [1].

Definition 2: Given a set of n-dimensional vectors $\beta,\alpha_1,\alpha_2,\ldots\alpha_m$, if there exists a set of scalars $k_1,k_2,k_3\ldots k_m$ such that $\beta=k_1\alpha_1+k_2\alpha_2\ldots+k_m\alpha_m$, then β is called a linear combination of $\alpha_1,\alpha_2,\ldots\alpha_m$, or β can be linearly expressed using $\alpha_1,\alpha_2,\ldots\alpha_m$ [2].

The linear dependence of vectors in space can be manifested in several relationships, as shown in Figure 1.

(1) When there is linear dependence, according to the definition, there exist non-zero coefficients k_1,k_2 such that $k_1\alpha+k_2\beta=0$. Assuming k_1 is not zero, we can express $\alpha=k_2/k_1\cdot\beta$. Thus, α,β are proportional, and their components are also proportional. For example, if $\alpha=(1,2)$ and $\beta=(2,4)$, the corresponding components of α,β are proportional. Therefore, a and b are linearly dependent. This can be illustrated on a graph in Figure 1(1), where the two vectors are col-linear, indicating the linear dependence between them.

(2) When α,β,γ are linearly dependent, according to the theorem for linear dependence, assuming γ can be expressed as a linear combination of α,β, we have $\gamma=k_1\alpha+k_2\beta$. Thus, γ is generated by the two vectors $k_1\alpha,k_2\beta$ using the parallelogram rule. Consequently, α,β and γ lie on the same plane, as shown in Figure 1(2).

(3) When α,β,γ, and ω are linearly dependent, at least one vector can be expressed as a linear combination of the remaining vectors. Assuming that ω can be expressed as a linear combination of α,β and γ, we have $\omega=k_1\alpha+k_2\beta+k_3\gamma$ ω, where k_3, and $\omega=k_1\alpha+k_2\beta+k_3\gamma$ are arbitrary real numbers. Thus, we can write ω as a linear combination of α, β, and γ, denoted by $L(\alpha,\beta,\gamma)$, as shown in Figure 1(3).

(4) For four or more vectors, at least one vector belongs to the subspace generated by the remaining vectors, following the same reasoning.

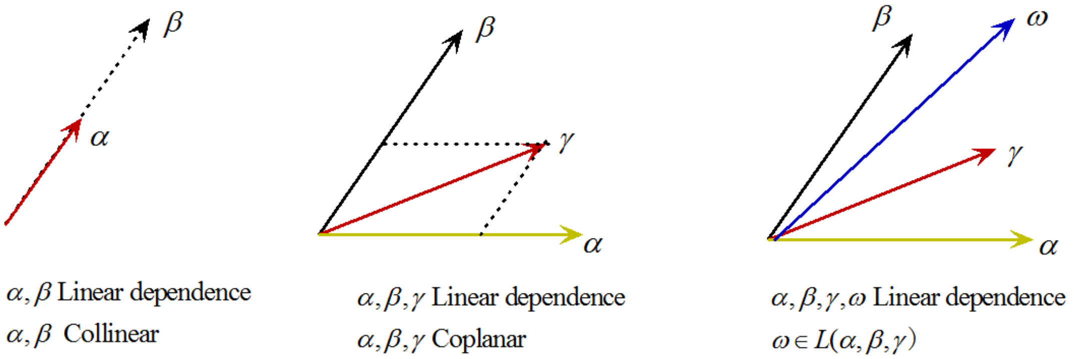

Figure 1: Geometric structure of linear dependence in space

3. Solutions of Non-Homogeneous Linear Systems

In engineering-oriented linear algebra textbooks, we have found that the treatment of the structural content of solutions to non-homogeneous linear equations is often overly simplified. Either only the theorem is presented without a proof, or there is a proof, but the understanding of the essence of the theorem is lacking. Without the aid of visual spatial results, the true essence of the theorem cannot be fully understood.

When teaching the structure of solutions to non-homogeneous linear systems, in addition to explaining the structural theorems of non-homogeneous linear equations, the corresponding spatial structures of the solutions should be emphasized. The teaching process can consider introducing examples first, then summarizing the content of the theorem, and finally presenting the proof of the theorem's approach.

Example 1: In Figure 1, given the line $x+y=0$, find the solution to $x+y=1$

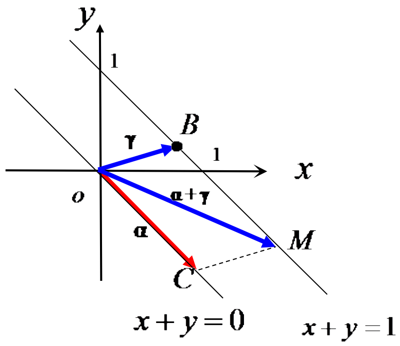

Figure 2: The relationship between solutions of a non-homogeneous linear system and the derived homogeneous system.

In Figure 2, let B be an arbitrary point on the line $x+y=1$, and let γ be the vector corresponding to point B, originating from the origin. Point C is another point on the line $x+y=0$, and its corresponding vector is α. Now, let M be a point on the line $x+y=1$ corresponding to the vector $\alpha+\gamma$, and let be the corresponding solution vector. As point C traverses the entire line $x+y=0$, the points to which the vector $\alpha+\gamma$ points will cover all the points on line $x+y=1$. Therefore, the solution to the non-homogeneous linear system $x+y=1$will consist of the general solution to the corresponding homogeneous system $x+y=0$ and any particular solution γ to the non-homogeneous system $x+y=1$.

Through the figure, we can observe that the solution vector M on line $x+y=1$ provides a full understanding of the structure of solutions to the corresponding homogeneous linear system. The geometric interpretation of solutions to the homogeneous linear system lies in the set of points on a line passing through the origin in a 2D plane. Therefore, all the points on line $x+y=0$ in the figure represent the solution set to a 2nd order homogeneous linear system. Meanwhile, all the points on line $x+y=1$ correspond to the solution set of a non-homogeneous 2nd order linear system.

Example 2: Find the solutions of a system of three linear equations $\begin{cases} x_1+x_2+4x_3=0 \\ -x_1+4x_2+x_3=0 \\ x_1-x_2+2x_3=0 \end{cases}$ and the corresponding non-homogeneous linear equations $\begin{cases} x_1+x_2+4x_3=4 \\ -x_1+4x_2+x_3=16 \\ x_1-x_2+2x_3=-4 \end{cases}$.

Solution: Simplify the augmented matrix of the non-homogeneous linear equations to row echelon form.

$\tilde{A}=\begin{pmatrix}1 & 1 & 4 & 4\\ -1 & 4 & 1 & 16\\ 1 & -1 & 2 & -4\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0 & 3 & 0\\ 0 & 1 & 1 & 4\\ 0 & 0 & 0 & 0\end{pmatrix}$, so $\begin{cases}x_1=-3x_3\\ x_2=-x_3+4\end{cases}$. Set $x_3=k$, we obtain the general solution of the system of equations $\begin{cases}x_1=-3x_3\\ x_2=-x_3+4\end{cases}$ as $X=k(-3,-3,1)^T$, where k is an arbitrary constant. One particular solution of the corresponding non-homogeneous linear equations is $\eta^*=k(0,4,0)^T$. We can examine the relationship between the solutions of the two sets of equations in the graph.

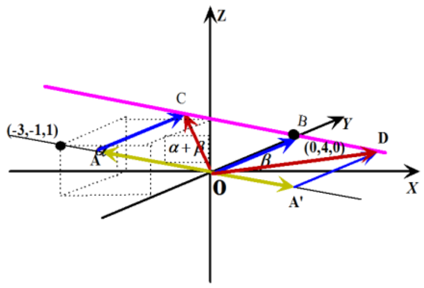

Figure 3: The relationship between solutions of a non-homogeneous system of three linear equations and the corresponding solutions of the associated homogeneous system

In Figure 3, the points on the line OA represent all the solutions of the homogeneous linear equations A. Let α be an arbitrary solution vector on the line OA, and let $\eta^* = (0, 4, 0)^T$ be a particular solution of the non-homogeneous linear equations. Then, $\gamma = \alpha + \eta^*$ corresponds to the point C. As the vector α varies over the points on the line OA, the point C will traverse all the points on the line BC. These points represent all the solutions corresponding to the set B. Therefore, the solutions of the non-homogeneous equations will be the sum of the general solution of the homogeneous linear equations and the particular solution of the non-homogeneous equations. Through the above two examples, we can clearly understand the relationship between the solutions of homogeneous and non-homogeneous linear equations. Therefore, the following summary can be made:

Theorem 1: If the linear equation system $AX=b$ has a solution, then one solution of the equation system $AX=b$ plus a solution of its associated homogeneous system is also a solution of the equation system $AX=b$. Any solution of the equation system $AX=b$ can be expressed as the sum of any particular solution of $AX=b$ and a solution of its associated homogeneous system.

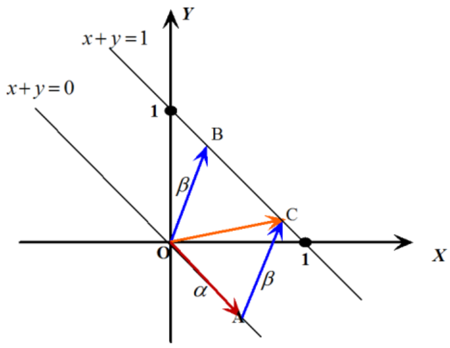

Figure 4: The solution structure of a non-homogeneous linear

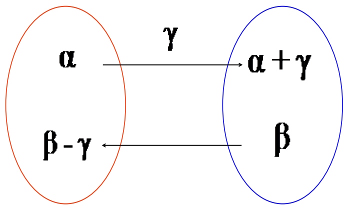

Figure 5: Illustrating the relationship between a non-homogeneous linear system of equations system of equations

To prove the theorem, let's consider using a graphical method to illustrate it. If $AX=b$ has a solution, then its particular solution γ plus any arbitrary solution α of the associated homogeneous system is still a solution of $AX=b$ (as shown in Figure 3). If $AX=b$ has a solution, then any solution β of the equation system minus the particular solution γ, i.e., $\beta-\gamma$, is a solution of the associated homogeneous system $AX=0$. By using a graphical representation, we can quickly understand the structure of solutions for non-homogeneous linear equation systems (as shown in Figure 4). Figure 5 reflects the relationship between the solution set of the non-homogeneous system and the solution set of the associated homogeneous system, where W_0 represents the solution set of $AX=0$, and W represents the solution set of $AX=b$.

4. Gram-Schmidt orthogonal formula

In engineering linear algebra textbooks, the Gram-Schmidt orthogonal formula is presented by providing the Gram-Schmidt orthogonal formula formula without explanation. This leads to students being unable to accurately understand the formula and can only memorize it. It would be helpful to understand the essence of Gram-Schmidt orthogonal formula by combining it with the spatial relationship between vectors.

The Gram-Schmidt orthogonal formula transforms any basis $\alpha_1,\alpha_2,\ldots\alpha_r$ of vector space $V(V\subset R^n)$ into an orthogonal normalized basis. The specific steps are as follows [3]:

First, take $e_1=\dfrac{\alpha_1}{|\alpha_1|}, e_i=\dfrac{\beta_1}{|\beta_1|}, \beta_i=\alpha_i-[\alpha_i,e_2]e_2-\ldots-[\alpha_i,e_{i-1}]e_{i-1}, i=2,3,\ldots,n.$ Then $\beta_1,\beta_2,\ldots\beta_r$ are a set of pairwise orthogonal bases of space V [3]. It is necessary to have a clear understanding of the geometric meaning of the dot product of two vectors before revealing the geometric meaning of the Gram-Schmidt formula. For the dot product $[\alpha_1,\alpha_2]$ of vectors α_1 and α_2, its result is a scalar. The value of $[\alpha_1,\alpha_2]$ is the length of the projection OA from vector α_2 to vector α_1 (as shown in Figure 6).

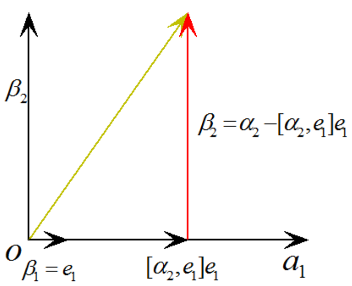

Figure 6

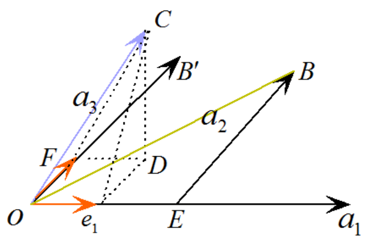

Figure 7

Figure 6: The geometric interpretation of vector dot Product Figure 7: The geometric interpretation of Gram-Schmidt orthogonal formula for 3 vectors

In engineering linear algebra textbooks, the Gram-Schmidt's orthogonal formula is presented by

(1) When $r=2, \beta_1 = e_1, \beta_2 = \alpha_2 - [\alpha_2, e_1]e_1$. From Figure 6, first shorten α_1 to a length of 1, resulting in vector $\beta_1 = e_1$, which is the projected vector $[\alpha_2, e_1]e_1$ from vector α_2 to e_1. Then β_2 is the vector. It can be seen from the figure that β_2 is perpendicular to α_1.

(2) When $r=3, \beta_1 = e_1, \beta_2 = \alpha_2 - [\alpha_2, e_1]e_1, \beta_3 = \alpha_3 - [\alpha_3, e_1]e_1 - [\alpha_3, e_2]e_2$. In Figure 7, first normalize α_1, point A is the projection point of vector α_2 terminal B onto vector e_1, and the vector is $[\alpha_2, e_1]e_1$. β_2 is the vector. Normalize β_2 to get e_2, draw a parallel vector through point O to the endpoint C of vector α_3, and draw a perpendicular line in the plane where e_1 and e_2 are located from point C with D as the foot. Then $\overrightarrow{CD} \perp e_1$, $\overrightarrow{CD} \perp e_2$, draw a perpendicular line from point C to e_1 with E as the foot, and draw a perpendicular line from point C to e_2 with F as the foot. The vector $\beta_3 = \alpha_3 - [\alpha_3, e_1]e_1 - [\alpha_3, e_2]e_2$ can be obtained. Thus, the linearly independent vector group $\alpha_1, \alpha_2, \alpha_3$ is transformed into the vector group $\beta_1 = \alpha_1, \beta_2 = \overrightarrow{AB}, \beta_3 = \overrightarrow{CD}$ normalized, resulting in a set of pairwise orthogonal unit vectors e_1, e_2, e_3.

5. Conclusion

Through the introduction of these three examples, it can be clearly seen that the definitions, theorems, and some methods in engineering linear algebra have geometric meanings. By demonstrating the spatial geometric relationships, one can understand the essence of these problems. This greatly helps to reduce the abstractness of textbooks and facilitate students' learning of engineering linear algebra.

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