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Equivalence of 2-tensors and 3-tensors under Local Unitary and General Linear Groups

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Abstract: In quantum information, pure states correspond to tensors. Two states equivalent under local unitary group or local general linear group can be used for the same quantum-information tasks, and it is thus an essential problem to determine whether two tensors are equivalent. In this paper, we determine the two sorts of equivalence for 2-tensors in terms of the Schmidt decomposition and singular value decomposition of matrices. We also extend the equivalence to some 3-tensors, proving that Schmidt decomposition is unachievable for 3-tensors. However, we present a method to express 3-tensors using the logic behind Schmidt decomposition. Furthermore, we generalized the scenario to higher dimensions and discussed the characteristic behaviors of different groups in tensor transformations.

1. Introduction

Quantum information science is an emerging and swiftly evolving field, with quantum entanglement being its main component of interest. Using local quantum operations and classical communication (LOCC), one can consolidate partially entangled pairs of particles into fewer pairs [1]. Under these same operations, classification of pure states of multipartite entanglement can be simplified using asymptotic transformations [2]. Two pure states of a bipartite system can be transformed locally, with the most efficient method identified by the authors in Ref [3]. A novel method to compute the entanglement of formation for a pair of qubits employs local quantum operations combined with classical communications (LQCC). This approach relates to the multiplication using a Lorentz matrix and incorporates a normalization [4]. Equivalence classes in sets of entangled states are defined using invertible local transformations. A *W* state is remarkable in that it retains bipartite entanglement for three qubits [5]. In Ref [6], authors found a classification methods for three qubits based on five entanglement parameters. In Ref [7], the authors examined the classification of four-qubit pure states using stochastic local operations and classical communications (SLOCC). The findings in Ref [8] expanded this classification to include multipartite states, encompassing infinitely many types of states that are inequivalent under SLOCC. Multidimensional determinants, also referred to as "hyperdeterminants" can be used to establish how local actions affect inequivalent multipartite entangled classes, providing classification of these states [9]. In Ref [10], the authors introduce a criterion for determining the feasibility of transforming between two pure states via SLOCC. This standard can also categorize $2 \times M \times N$ systems. In Ref [11], unitary equivalence of quantum states is conditioned using the von Neumann entropy. This also uses the partial trace operation, a generalisation of the probability concept of marginal distributions for multi-partite quantum systems, and the strong subadditivity (SSA) property, relating von Neumann entropies of quantum systems, are explained in linear algebra terms in Ref [12]. In Ref [13], authors introduce the Greenberger-Horne-Zeilinger (GHZ) states. Furthermore, Ref [5] proves that GHZ states and W states are not mutually convertible under SLOCC, as their minimal product decompositions have different numbers of terms.

In this paper, we present applications of the Kronecker products on high-dimension tensors. We start by tackling the following problem. We construct two 2-tensors,

$T_1 = a\,|0> \otimes |0> + \, b\,|1> \otimes |1>$, and $T_2 = c\,|0> \otimes |0> + \, d\,|1> \otimes |1>$, where $|0> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and a, b, c, d are complex numbers. Can we find an element U in $U(2) \times U(2)$ such that $UT_1 = T_2$, where $U(2)$ is the 2×2 unitary group? If yes, then we say that the two 2-tensors are equivalent under local unitary (LU) transformation. If the group $U(2)$ is replaced by the $GL(2)$, i.e. the 2×2 general linear group, then we say that the two 2-tensors are equivalent under SLOCC transformation. For this purpose, we introduce the Schmidt decomposition for 2-tensors, the singular value decomposition (SVD) for arbitrary matrices, and construct the explicit conditions by which T_1 and T_2 are equivalent under LU and SLOCC transformations. We also generalize the problem to higher-dimension 2-tensors and 3-tensors. In particular, we find some 3-tensors which are not equivalent under LU transformation.

2. Preliminaries to Matrices

In this section, we introduce the preliminary knowledge and facts used in this paper. In Sec. 2.1, we introduce the different types of matrices. In Sec. 2.2, we review some basic matrix operations, namely addition, multiplication, and exponentiation. In Sec. 2.3, we present matrix transposition. In Sec. 2.4, we discuss the unitary matrix. In Sec. 2.5, we examine symmetric matrices. In Sec. 2.6, we introduce the Kronecker product. In Sec. 2.7, we present singular value decomposition (SVD), and in Sec. 2.8, Schmidt decomposition.

2.1 Types of Matrices

We refer to the row matrix as $(a_1 \quad a_2 \quad \cdots \quad a_n)$.

We refer to the column matrix as $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$.

We refer to the zero matrix as $\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$.

We refer to the diagonal matrix as $\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_2 \end{pmatrix}$.

We refer to the identity matrix (also denoted as I) as $\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$.

2.2 Basic Matrix Operations

It is only possible to add matrices of the same size, i.e. two matrices A and B can only be added if

$A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$. Then, we have $A + B = C = (c_{ij})_{m\times n}$ where

$$c_{ij} = a_{ij} + b_{ij}. \tag{1}$$

Matrix addition satisfies the following properties:

$$A + B = B + A \tag{2}$$

$$A + (B + C) = (A + B) + C \tag{3}$$

When performing matrix multiplication, the first matrix's columns should match the number of rows in the second matrix. For $A = (a_{ij})_{m\times s}$ and $B = (b_{ij})_{s\times n}$, we have $AB = C = (c_{ij})_{s\times s}$ where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{is}b_{sj} = \sum_{k=1}^{s} a_{ik}b_{kj}\,. \tag{4}$$

Matrix multiplication satisfies the following properties:

$$(AB)C = A(BC) \tag{5}$$

$$\lambda(AB) = (\lambda A)B \tag{6}$$

$$A(B + C) = AB + AC, (B + C)A = BA + CA \tag{7}$$

$$E_m A_{m\times n} = A_{m\times n}E_n = A_{m\times n} \tag{8}$$

$$\lambda E \text{ is a scalar matrix} \tag{9}$$

Matrix exponentiation can only be applied to square matrices. For $= (a_{ij})_{s\times s}$, we have

$$A^k = \underbrace{AA \ldots A}_{k} \tag{10}$$

and

$$A^k A^l = A^{k+l}, (A^k)^l = A^{kl}\,. \tag{11}$$

The following properties also hold when $AB = BA$:

$$(AB)^k = A^k B^k \tag{12}$$

$$(A + B)^2 = A^2 + 2AB + B^2 \tag{13}$$

$$(A + B)(A - B) = A^2 - B^2 \tag{14}$$

2.3 Matrix Transposition

Matrix transposition consists of switching the rows and columns of a matrix and is denoted by ^{T}. For example, for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.

Matrix transposition satisfies the following properties:

$$(A^T)^T = A \tag{15}$$

$$(A + B)^T = A^T + B^T \tag{16}$$

$$(\lambda A)^T = \lambda A^T \tag{17}$$

$$(AB)^T = B^T A^T \tag{18}$$

2.4 Unitary Matrices

We denote U as the unitary matrix. The unitary matrix satisfies $U \cdot U^{\dagger} = 1_n$, where $U^{\dagger} = (U^*)^T$ and U^* denotes the complex conjugate matrix of U.

2.5 Symmetric Matrices

A square matrix $A = (a_{ij})_{n \times n}$ is said to be symmetric if $A = A^T$, i.e. $a_{ij} = a_{ji}$ for i,j = 1, 2, ... n. On the other hand, A is said to be asymmetric/antisymmetric/skew symmetric if $A = -A^T$. In this case, all numbers on the diagonal must be 0.

A Hermitian matrix is a square matrix $A = (a_{ij})_{n \times n}$ such that $a_{ij} = a_{ji}{}^*$ for any i,j.

Here is an interesting problem using symmetric matrices:

Given the row matrix $X = (x_1, x_2, \dots, x_n)^T$ satisfying $X^T X = 1$, and the identity matrix of size n, E, if we define the matrix $H = E - 2XX^T$, prove that H is symmetric and that $HH^T = E$.

The proof goes as follows:

$$H^T = (E - 2XX^T)^T = E^T + (-2XX^T)^T = E - 2(XX^T)^T$$
$$= E - 2(X^T)^T X^T = E - 2XX^T = H$$
$$HH^T = H^2 = (E - 2XX^T)^2 = E^2 - 4XX^T + (-2XX^T)^2$$
$$= E - 4XX^T + 4XX^T XX^T = E - 4XX^T + 4X(X^T X)X^T$$
$$= E - 4XX^T + 4XX^T = E.$$

2.6 Kronecker Product

The Kronecker product, sometimes referred to as the direct product or tensor product, is a method for multiplying matrices that don't meet the criteria for standard matrix multiplication.

Given matrix A of size $n \times p$ and matrix B of size $m \times q$, their resultant matrix will have dimensions $mn \times pq$ and is computed as follows:

$$A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,p}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,p}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,p}B \end{bmatrix} \tag{19}$$

The Kronecker product satisfies the following properties :

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C \tag{20}$$
$$A \otimes (B + C) = (A \otimes B) + (A \otimes C) \tag{21}$$
$$a \otimes A = A \otimes a = aA, \text{where } a \text{ is a scalar} \tag{22}$$
$$(A \otimes B)(C \otimes D) = AC \otimes BD, \text{for conforming matrices} \tag{23}$$

2.7 Singular Value Decomposition (SVD)

A decomposition of the matrix $X = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \cdot V$ is called the singular value decomposition (SVD) of X, where $D = \mathrm{diag}(d_1, \dots, d_r)$, with $d_1 \geq \cdots \geq d_r > 0$ as the singular values of X. The singular values are uniquely determined by X. This means that, given a matrix X, it can be written as

$X = U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \cdot V = U_1\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \cdot V_1$ with $D = \mathrm{diag}(d_1, \dots, d_r)$ and $D_1 = \mathrm{diag}(f_1, \dots, f_r)$, where $d_1 \geq \cdots \geq d_r > 0$ and $f_1 \geq \cdots \geq f_r > 0$, $d_i = f_i$ for $0 \leq i \leq r$.

2.8 Schmidt Decomposition

For a matrix $M = [|a_1>, |a_2>, \dots, |a_n>]^{m\times n}$, where $|a_1> \in C^m$, if $\sum_{j=0}^{n} c_j |a_j> = 0$, $c_j \in C$, then $c_j = 0$ for all j, and $|a_1>$, $|a_2>$, …, $|a_n>$ are linearly independent, and this property determines the rank r of a matrix.

Every 2-tensor T has a Schmidt decomposition, i.e.

$$T = \sum_{j=1}^{r} \sqrt{c_j}\, |a_j> \otimes |b_j> \tag{24}$$

where $c_j > 0$, $|a_j>$'s are orthonormal vectors and $|b_j>$'s are orthonormal vectors.

Using SVD and given $<b_j| = (|b_j>)^\dagger$ and $<b_j{}^*| = (|b_j>)^T$, we can write

$$T' = U\begin{pmatrix} \sqrt{c_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{c_r} \end{pmatrix} V = U\left(\sum_{j=1}^{r} \sqrt{c_j}\, |j><j|\right) V$$

$$= \sum_{j=1}^{r} \sqrt{c_j}\, |a_j> \otimes <b_j{}^*| = \sum_{j=1}^{r} \sqrt{c_j}\, U\, |j><j|\, V.$$

3. Results

Within this section, we present the primary findings of our study. In Sec. 3.1, we apply the Kronecker product to 2-tensors in $C^2 \otimes C^2$. Then, in Sec. 3.2, we generalize the conclusion to higher-dimension tensors. In Sec. 3.3, we show that a Schmidt decomposition for a 3-tensor is impossible to achieve. In Sec. 3.4, we follow with a generalization of the question in 3.1 to higher-dimension tensors. In. Sec. 3.5, we discussed the two groups, $GL(2,2,2)$ and $U(4) \times U(2)$, illustrating how they transform distinct tensors.

3.1 Applications of the Kronecker Product on 2-Tensors

We can consider the following problem to better understand the use of the Kronecker product:

We construct two 2-tensors,

$$T_1 = a\, |0> \otimes |0> + \; b\, |1> \otimes |1>, \text{ and}$$

$$T_2 = c\, |0> \otimes |0> + \; d\, |1> \otimes |1>,$$

where $|0> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and a, b, c, d are complex numbers.

Can we find an element U in $U(2) \times U(2)$ such that $UT_1 = T_2$, where $U(2)$ is the 2×2 unitary group? If yes, we say that T_1 and T_2 are equivalent under local unitary (LU) group.

Let $U = V \otimes W$, where V and W are elements of $U(2)$.

We can look at the norm of each side of the equation

$$UT_1 = T_2. \tag{25}$$

$$\|RHS\| = \sqrt{|c|^2 + |d|^2}$$

$$\|LHS\| = \left\| (V \otimes W) \begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix} \right\| = \sqrt{(a^* \quad 0 \quad 0 \quad b^*) \begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix}} = \sqrt{|a|^2 + |b|^2}$$

This gives us the equation

$$|a|^2 + |b|^2 = |c|^2 + |d|^2. \tag{26}$$

It is possible to prove a bijection (an isomorphism between two sets) between elements of the following two sets:

$$\phi = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{ij} |i\rangle \otimes |j\rangle \epsilon\, C^m \otimes C^n$$

$$\psi = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{ij} |i\rangle \otimes \langle j| \,\epsilon\, C^{m\times n}.$$

Using the norm obtained for the RHS, we have that T_2 corresponds to $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$.

Developing on Equation (25) using the previously determined conditions, we obtain $aV|0\rangle \otimes W|0\rangle + bV|1\rangle \otimes W|1\rangle$, which corresponds to the matrix

$$aV|0> \otimes (W \mid 0>)^T + bV|1> \otimes (W \mid 1>)^T$$

$$= aV|0> \otimes <0|W^T + bV|1> \otimes <1|W^T$$

$$= V(a|0><0| + b|1><1|)W^T$$

$$\Leftrightarrow V \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} W^T = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}.$$

From this, and using SVD, we observe that

$$\text{LHS} = V \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} W^T$$

$$= V \begin{pmatrix} |a|e^{i\alpha} & 0 \\ 0 & |b|e^{i\beta} \end{pmatrix} W^T$$

$$= V \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} |a| & 0 \\ 0 & |b| \end{pmatrix} W^T$$

$$\text{RHS} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} |c|e^{i\gamma} & 0 \\ 0 & |d|e^{i\delta} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} |c| & 0 \\ 0 & |d| \end{pmatrix} 1_2.$$

Hence, the necessary condition for the existence of a matrix U satisfying the problem is that $|a| = |c|$ and $|b| = |d|$ or $|a| = |d|$ and $|b| = |c|$.

If $|a| = |c|$ and $|b| = |d|$, we can write

$$T_1 = |a|e^{i\alpha}\,|0\rangle \otimes |0\rangle + |b|e^{i\beta}\,|1\rangle \otimes |1\rangle, \text{ and}$$

$$T_2 = |a|e^{i\gamma}\,|0\rangle \otimes |0\rangle + |b|e^{i\delta}\,|1\rangle \otimes |1\rangle.$$

The unitary matrix U which can convert T_1 to T_2 can be written as

$$U = V \otimes W = \begin{pmatrix} e^{i(\gamma-\alpha)} & 0 \\ 0 & e^{i(\delta-\beta)} \end{pmatrix} \otimes I_2. \tag{27}$$

Similarly, if $|a| = |d|$ and $|b| = |c|$, the unitary matrix U which can convert T_1 to T_2 can be written as

$$U = V \otimes W = \begin{pmatrix} 0 & e^{i(\delta-\alpha)} \\ e^{i(\gamma-\beta)} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{28}$$

To conclude, we have presented the conditions by which two 2-tensors T_1 and T_2 are LU equivalent.

3.2 Applications of the Kronecker Product on 2-Tensors with Higher Dimensions

It is possible to generalize the question in Sec. 3.1 into finding a unitary matrix U in $U(d) \times U(d)$, where $U(d)$ is the $d \times d$ unitary group, satisfying the condition that $UT_1 = T_2$, where

$$T_1 = a_0|0> \otimes|0> + a_1|1> \otimes|1> + \ldots + a_{d-1}|d-1> \otimes|d-1>, \text{ and}$$

$$T_2 = b_0|0> \otimes|0> + b_1|1> \otimes|1> + \ldots + b_{d-1}|d-1> \otimes|d-1>.$$

Following the reasoning for the $U(2) \times U(2)$, we can conclude that

$$\{|a_0|, \ldots, |a_{d-1}|\} \leftrightarrow \{|b_0|, \ldots, |b_{d-1}|\}.$$

We can rewrite T_1 and T_2 as such:

$$T_1 = |a_0|e^{i\alpha_0}\,|0\rangle \otimes |0\rangle + \cdots + |a_{d-1}|e^{i\alpha_{d-1}}\,|d-1\rangle \otimes |d-1\rangle, \text{ and}$$

$$T_2 = |b_0|e^{i\beta_0}\,|0\rangle \otimes |0\rangle + \cdots + |b_{d-1}|e^{i\beta_{d-1}}\,|d-1\rangle \otimes |d-1\rangle.$$

If we consider the case where $|a_i| = |b_i|$ for $0 \le i \le d-1$, we can write U as

$$U = V \otimes W = \begin{pmatrix} e^{i(\beta_0-\alpha_0)} & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & e^{i(\beta_{d-1}-\alpha_{d-1})} \end{pmatrix} \otimes I_d. \tag{29}$$

If we consider the case where $|a_i| = |b_{\sigma(i)}|$ for $0 \le i \le d-1$, we define P such that

$$(P \otimes P)T_1 = a_0|\sigma(0)\rangle \otimes |\sigma(0)\rangle + \ldots + a_{d-1}|\sigma(d-1)\rangle \otimes |\sigma(d-1)\rangle.$$

We want to find U such that $U(P \otimes P)T_1 = T_2$.
Then, we can write U as

$$U = V \otimes W = \begin{pmatrix} e^{i(\beta_{\sigma(0)}-\alpha_0)} & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & e^{i(\beta_{\sigma(d-1)}-\alpha_{d-1})} \end{pmatrix} \otimes I_d\,. \tag{30}$$

We denote as r and s the Schmidt ranks of the 2-tensors T_1 and T_2, respectively.
We can write

$$T_1 = (U_1 \otimes V_1) \sum_{j=1}^{r} \sqrt{c_j}\, |j\rangle \otimes |j\rangle$$

$$T_2 = (U_2 \otimes V_2) \sum_{i=1}^{s} \sqrt{d_i}\, |i\rangle \otimes |i\rangle$$

with $c_1 \geq \cdots \geq c_r > 0$ and $d_1 \geq \cdots \geq d_s > 0$.
It then suffices to compare the Schmidt coefficients of the two tensors.
Let

$$|W\rangle \neq (A \otimes B \otimes C)(|0,0,0\rangle + |1,1,1\rangle) \quad (31)$$

where $|0,0,0\rangle + |1,1,1\rangle = |GHZ\rangle$.
If $|\Psi\rangle \in C^2 \otimes C^2 \otimes C^2$, then $\exists$ $A \otimes B \otimes C \in GL(2,2,2) = GL(2) \times GL(2) \times GL(2)$

such that $(A \otimes B \otimes C)\, |\Psi\rangle = \begin{cases} |0,0,0\rangle \\ \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)|0\rangle \\ |W\rangle \\ |GHZ\rangle \\ \frac{1}{\sqrt{2}}|0,0,0\rangle + |1,0,1\rangle \\ \frac{1}{\sqrt{2}}|0\rangle(|0,0\rangle + |1,1\rangle) \end{cases}$.

Here, $GL(d)$ is the set of all $d \times d$ invertible matrices, $U(d)$ is the set of all $d \times d$ unitary matrices, so $U(d)$ is a proper subset of $GL(d)$.

We can see that

$$(A \otimes B \otimes C)|0,0,0\rangle = A|0\rangle \otimes (B|0\rangle \otimes C|0\rangle) = |0,0,0\rangle + |1,1,0\rangle \in H_{abc}.$$

Furthermore, we know that $U(d) \subset GL(d)$. Let $X = (DU_1{}^{\dagger} \otimes V_1{}^{\dagger}), Y = U_2F \otimes V_2$, where $D = diag(\sqrt{c_1{}^{-1}}, \ldots, \sqrt{c_r{}^{-1}}), F = diag(\sqrt{d_1}, \ldots, \sqrt{d_r})$. Then,

$$XT_1 = \sum_{j=1}^{r} |j\rangle \otimes |j\rangle$$

$$\Rightarrow YXT_1 = (U_2 \otimes V_2) \sum_{j=1}^{r} \sqrt{d_j}\, |j\rangle \otimes |j\rangle = T_2.$$

Knowing $YX = U_2FDU_1{}^{\dagger} \otimes V_2V_1{}^{\dagger} \in GL(r) \times GL(r)$, we have

$$T_1 = 0.6|0,0\rangle + 0.8|1,1\rangle$$

$$\text{and } T_2 = \frac{1}{\sqrt{2}}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle + \frac{1}{\sqrt{2}}\left(\frac{|2\rangle + |3\rangle}{\sqrt{2}}\right)|1\rangle.$$

Thus, we can rewrite a tensor in $GL(2,2,2)$ in the form $U(4) \times U(2)$.

3.3 Schmidt Decomposition for 3-Tensors

If $|\psi\rangle \in C^m \otimes C^n$, then $|\psi\rangle \in \text{span}\{|i\rangle \otimes |j\rangle\}_{j=0,\ \ldots,\ n-1}^{i=0,\ \ldots,\ m-1}$, which means that we can write $|\psi\rangle = \sum_{i,j} c_{ij}\, |i\rangle \otimes |j\rangle$, where c_{ij} is a complex number.

For a 3-tensor $|\psi> \in C^m \otimes C^n \otimes C^p$, we can then write

$$|\psi> \in \text{span}\{|i> \otimes |j> \otimes |k>\}_{\substack{i=0, \ldots, m-1 \\ j=0, \ldots, n-1 \\ k=0, \ldots, p-1}}. \quad (32)$$

For example, we consider the 3-tensor

$$|W> = |0> \otimes |0> \otimes |1> + |0> \otimes |1> \otimes |0> + |1> \otimes |0> \otimes |0> \in C^2 \otimes C^2 \otimes C^2$$
$$= |0> \otimes (|0> \otimes |1> + |1> \otimes |0>) + |1> \otimes (|0> \otimes |0>)$$
$$= \sqrt{2}\,|0> \otimes \frac{|0> \otimes |1> + |1> \otimes |0>}{\sqrt{2}} + |1> \otimes (|0> \otimes |0>). \quad (33)$$

We claim that the decomposition in (33) is the Schmidt decomposition for |W> taken as a 2-tensor in the bipartite space being the tensor product of C^2 and $C^2 \otimes C^2$. We compute that

$$(<0| \otimes <0|)\left(\frac{|0> \otimes |1> + |1> \otimes |0>}{\sqrt{2}}\right) \quad (34)$$
$$= (<0| \otimes <0|)\frac{|0> \otimes |1>}{\sqrt{2}} + (<0| \otimes <0|)\frac{|1> \otimes |0>}{\sqrt{2}}$$
$$= 0 + 0 = 0.$$

Furthermore, we show that

$$\left(\frac{<0| \otimes <1| + <1| \otimes <0|}{\sqrt{2}}\right)\left(\frac{|0> \otimes |1> + |1> \otimes |0>}{\sqrt{2}}\right)$$
$$= \frac{1}{2}(<0,1| \cdot |0,1> + <0,1| \cdot |1,0> + <1,0| \cdot |0,1> + <1,0| \cdot |1,0>)$$
$$= \frac{1}{2}(1+1) = 1$$

and $(<0| \otimes <0|)(|0> \otimes |0>) = 1$.

Hence, the two vectors $|0> \otimes |0>$ and $\frac{|0> \otimes |1> + |1> \otimes |0>}{\sqrt{2}}$ are orthonormal. Therefore, we have proven the claim below (33).

On the other hand, we can write $|W>$ in a way similar to the Schmidt decomposition for 2-tensors. For example, we assume the decomposition

$$|W> = \sum_{j=1}^{r} \sqrt{d_j}\,|a_j, b_j, c_j>, \quad (35)$$

where $|a_j>$'s are orthonormal vectors in C^2, $|b_j>$'s are orthonormal vectors in C^2, and $|c_j>$'s are orthonormal vectors in C^2. In the following steps, we show that (35) does not hold. Since $|a_1>, \ldots, |a_r> \in C^2$, we have $r \leq 2$, giving only two cases for $|W>$, namely r = 1 and r = 2.

If $r = 1$, then (35) implies that $|W> = \sqrt{d}\,|a, b, c>$. Let $|a> = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$, $|b> = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$, and $|c> = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$. Then, from (35), we have

$$\sqrt{d}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \otimes \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = |0, 0, 1> + |0, 1, 0> + |0, 0, 1>$$

$$\Leftrightarrow \begin{bmatrix} a_0b_0c_0 \\ a_0b_0c_1 \\ a_0b_1c_0 \\ a_0b_1c_1 \\ a_1b_0c_0 \\ a_1b_0c_1 \\ a_1b_1c_0 \\ a_1b_1c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Looking at the 2nd and 6th entries, we can determine that $a_1 = 0$. Then, it is impossible for the 5th entry to equal 1. We have excluded the case r = 1 for (35).

It remains to consider the case r = 2 for (35). For this purpose, we firstly show the following fact.

Let $T = |0, 1> + |1, 0> + x\ |0, 0>$, where $x \epsilon C$ and $|y>$ be a non-zero matrix. We want to show that $(<y| \otimes 1)\ T \neq 0$.

$$(<y| \otimes 1)\ T$$
$$= (<y| \otimes 1)\ [a_1\ |\alpha_1> \otimes |\beta_1> + a_2\ |\alpha_2> \otimes |\beta_2>]$$
$$= a_1 <y\ |\alpha_1> \cdot |\beta_1> + a_2 <y\ |\alpha_2> \cdot |\beta_2>. \quad (36)$$

We rewrite $a_1 <y\ |\alpha_1>$ as x_1 and $a_2 <y\ |\alpha_2>$ as x_2. So for $(<y| \otimes 1)\ T$ to equal 0, $x_1 = x_2 = 0$, which is a contradiction.

Now we return to the case r = 2 for (35). We have

$$|\alpha\rangle = \sqrt{d_1}|a_1, b_1, c_1\rangle + \sqrt{d_2}|a_2, b_2, c_2\rangle. \quad (37)$$

We consider the case where $|a_1> = x\ |a_2>$. This means that $\exists\ |a_3> \in C^2/\{0\}$ s.t. $|a_3> \perp |a_2>$. Then, we obtain

$$(<a_3| \otimes 1_2 \otimes 1_2)(|0, 0, 1> + |0, 1, 0> + |0, 0, 1>) \quad (38)$$
$$= <a_3|0> \cdot |0, 1> + <a_3|0> \cdot |1, 0> + <a_3|1> \cdot |0, 0>$$
$$= 0$$

by multiplying by $(<a_3| \otimes 1_2 \otimes 1_2)$ on both sides of (37). Therefore, $|a_1> \neq x\ |a_2>$. We obtain that $\exists\ |x> \in C^2/\{0\}$ s.t. $|x> \perp |a_1>$. Multiplying by $(<x| \otimes 1_2 \otimes 1_2)$ on both sides of (35) gives

$$< x|0 > \cdot|0,1 > + < x|0 > \cdot|1,0 > + < x|1 > \cdot|0,0 >= \sqrt{d_2} < x|a_2 >|b_2, c_2 >$$
$$\Leftrightarrow < x \mid 0 > \left(|0,1 > + |1,0 > + \frac{< x \mid 1 >}{< x \mid 0 >} \mid 0,0 > \right) = 0$$

and from (36), this results in a contradiction. Therefore, a Schmidt decomposition of the 3-tensor $|W>$ is impossible to achieve. One can similarly show that the Schmidt decomposition for a generic 3-tensor does not exist.

3.4 Generalization to Higher Dimensions

If $\psi \in C^2 \otimes C^3 \otimes C^{N-1}$, then we can write $\psi = \sum_{j=0}^{1} \sum_{k=0}^{2} \sum_{l=0}^{N-1} c_{jkl}\ |i> \otimes |k> \otimes |l>$. We want to determine whether $\exists\ U \in GL(2) \times GL(3) \times GL(5)$ s.t. $U \cdot \psi = |0, 2, 4> + |0, 0, 1> + |0, 1, 1> + |1, 0, 2> + |1, 1, 3>$.

Let $T_1 = \sum_{j=0}^{r} \sqrt{d_j}\, |j> \otimes |j>$. Then, there exists $U \otimes r_i \subseteq GL(m)$ such that $(U \otimes V)\, T_1 = \sum_{j=0}^{r} |j> \otimes |j>$. Suppose $T_2 \subset C^m \otimes C^n$. There exists a matrix $X \subset U(m) \otimes GL(n)$ such that $XT_2 = \sum_{j=0}^{r} |j> \otimes |j>$, where r is the Schmidt rank of T_2.

Let $X = U \otimes V$. Then, we can write $X(|0,2,4> + |0,0,1> + |0,1,1> + |1,0,2> + |1,1,3>) = |0,2,4> + |1,2,1> + |0,0,1> + |0,1,1> + |1,0,2> + |1,1,3>$.

We have that

$$X_1 T_2 = \sum_{j=1}^{r} |j> \otimes |j> = X_1 T_2{}' \\ \Longrightarrow X_1 T_2 = X_1 T_2' \\ \Longrightarrow T_2 = X_1{}^{-1} X_1 T_2'. \tag{39}$$

Let $M_1 = |0,2><4| + |0,0><0| + |0,1><1| + |1,0><2| + |1,1><3|$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $M_2 = |0,2><4| + |1,2><1| + |0,0><0| + |0,1><1| + |1,0><2| + |1,1><2|$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Following the previous calculations, we can see that $\text{rank}(M_1) = 5$ and $\text{rank}(M_2) = 5$.

3.5 Properties of $GL(2,2,2)$ and $U(4) \times U(2)$

In Sec. 3.2, we have mentioned the two following groups: $GL(2,2,2)$ and $U(4) \times U(2)$.

In $GL(2,2,2)$, $|0,0,0>$ cannot be transformed into $\frac{|0,0,0>+|1,1,0>}{\sqrt{2}}$. However, we know that if $X \in U(4)$, then $X \cdot X^\dagger = 1_4$. Then, if $X|0,0> = \frac{|0,0>+|1,1>}{\sqrt{2}}$, then $(X \otimes 1_2)|0,0,0> = \frac{|0,0,0>+|1,1,0>}{\sqrt{2}}$.

On the other hand, in $U(4) \times U(2)$, $\frac{|0,0,0>+|1,1,0>}{\sqrt{2}}$ cannot be rewritten as $0.6|0,0> + 0.8|1,1>$. However, $X \in GL(2,2,2)$ such that $X = 1_2 \otimes 1_2 \otimes \begin{pmatrix} 0.6\sqrt{2} & 0 \\ 0 & 0.8\sqrt{2} \end{pmatrix}$ satisfies the condition.

4. Conclusion

We have established two types of equivalences for 2-tensors by applying local unitary groups and general linear groups. This equivalence is further extended to include certain 3-tensors. However, due to the lack of Schmidt decomposition in 3-tensors (or more generally, n-tensors when n > 2), this extension becomes significantly complex. A key question arising from this study is to explore how the number of parameters in 3-tensors can be reduced under these two established equivalences. We also attempt to expand the discussion on this issue to higher dimensions, discussing the distinct characteristics displayed by specific groups in this process. Furthermore, it remains to be determined whether the unitary groups spanning two systems can eliminate more parameters than the multiplication of two general linear groups.

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