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A Multivariate Statistical Process Control Model Based on CRITIC Entropy Method and EWMA

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Abstract: Statistical process control is a technique to monitor product or service quality timely that ensure stability. It promotes quality assurance, resource optimization, and is crucial to informed decision-making. Given the diversification of quality indicators, this paper introduces a multivariate EWMA control chart model based on the CRITIC and entropy weighting method. This model allows lack of knowledge of variable distributions and considers variable correlations, which demonstrates strong sensitivity to slight drifts in mean and volatility, even with a non-diagonal covariance matrix. Simulation experiments confirm its ability to identify process changes and their types by manipulating the mean vector and covariance matrix in five controlled experiments.

1. Introduction

Statistical Process Control (SPC) refers to the statistical techniques to assess and monitor various stages of a process. It aims at establishing and maintaining the process at an acceptable and stable level to ensure that products and services meet specified requirements. By monitoring real-time product data and using control charts, SPC can detect process faults promptly and predict potential variations, which helps producers optimize production parameters, processes, and technologies to ensure product quality, improve efficiency, and reduce costs. Specifically, Multivariate Statistical Process Control (MSPC) is necessarily required in a process that contains multiple quality indicators.

Since its introduction by Dr. Shewhart in the 1920s, Statistical Process Control has evolved. In the 1930s, Kendrick and Harry Romano proposed mean control charts and range control charts for controlling continuous data. Traditional Shewhart control charts include X-R charts [1][2] and X-S charts [3], limited to single characteristic controlling. In the 1950s, E.S. Page introduced the Cumulative Sum (CUSUM) control chart [4], and Robert A. Ross proposed the Exponentially Weighted Moving Average (EWMA) control chart [5][6][7], which demonstrate better sensitivity and responsiveness in simultaneous cases. Adaptive control charts [8][9], proposed in the 1970s by David D. MacCalister, can automatically adjust control limits according to data changes to adapt to different production environments. Khoo [10] proposed using median control charts instead of traditional mean control charts to address the issue of potential outliers or individual exceptional values in practical processes. The rapid development of modern manufacturing requires increasingly more application of multivariate control chart which is constructed in mainly two ways: first, constructing multivariate statistics and monitor according to the distribution of multiple variables, such as Hotelling T2 control charts [11][12]; second, applying dimension reduction approaches like PCA [13], ICA [14], and weighting methods [15][16]. There have been abundant SPC research achievements whereas the following aspects need further research: limited types of multivariate control charts, lack of sensitivity to detect subtle changes, high requirements for given information of distribution, and incomplete consideration of correlations between variables in existing dimension reduction methods.

In this paper, a new approach for constructing multivariate control charts is proposed. The approach involves two stages of dimension reduction to transform multiple indicators process control into univariate process: firstly, the CRITIC weighting method and secondly the entropy weighting method. Finally, an Exponentially Weighted Moving Average (EWMA) control chart is plotted. The innovation of this method lies in considering both the mean level and volatility of variables, as well as considering correlations between variables using the CRITIC weighting method. Five sets of training and testing control experiment simulations were conducted to validate the proposed approach, all demonstrating ideal monitoring sensitivity.

2. Theory and Method

2.1 CRITIC Entropy Weighting Method

The CRITIC weighting method is an objective weighting method [16] proposed by Danae Diakoulaki, based on the contrast intensity and confliction between variables. The larger the differences among the sample values of a certain indicator are, the more effective the indicator is. In other words, larger standard deviation evolves larger corresponding weight. Besides, if the confliction between a certain indicator and other indicators is small, i.e., the correlation coefficient is large, it indicates that the information reflected by the indicator overlaps with other indicators. Thus, the corresponding weight is smaller. The following is the calculation process of the CRITIC weighting method.

Assume that there are n samples and p detection indicators in a certain process, and the original data matrix is:

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{np} \end{pmatrix} \tag{1}$$

In the equation, x_{ij} represents the detection value of the j^{th} indicator for the i^{th} sample. The original data is normalized, and the positive and negative indicators are normalized using the following methods:

$$x'_{ij} = \frac{x_{ij}-x_{min}}{x_{max}-x_{min}}, x'_{ij} = \frac{x_{max}-x_{ij}}{x_{max}-x_{min}} \tag{2}$$

Where x_{min} and x_{max} represent the minimum and maximum values of the sample data in the same indicator. The standard deviation of the normalized data for the j^{th} indicator is calculated as:

$$\sigma_j = \sqrt{\frac{\sum_{i=1}^{n}\left(x'_{ij}-\bar{x}'\right)^2}{n}} \tag{3}$$

The correlation coefficient between the i^{th} and j^{th} indicators is calculated as:

$$r_{ij} = \frac{Cov\left(x_i', x_j'\right)}{\sqrt{\sigma_i \sigma_j}} \tag{4}$$

Where $x_i{}'$ and $x_j{}'$ represent the normalized data vectors for the i^{th} column and j^{th} column, respectively. Using the correlation coefficient, the confliction between the j^{th} indicator and other indicators is calculated, and then the information content of the j^{th} indicator is obtained using the CRITIC weighting method:

$$C_j = \sigma_j \sum_{j=1}^{p} \left(1 - r_{ij}\right) \tag{5}$$

Finally, the objective weight is calculated as:

$$W_j' = \frac{C_j}{\sum_{j=1}^{p} C_j} \tag{6}$$

The entropy method is another objective weighting method, based on the principle that the information content is negatively correlated with the probability of an event occurring. Conversely, the greater the variability or the uncertainty of an indicator, the more information it contains, and therefore, the larger the corresponding weight. The following is the calculation process of the entropy method.

First, normalize the indicators, following the steps described above, to obtain the normalized data x_{ij}'. Calculate the proportion of the i^{th} sample data under the j^{th} indicator:

$$\eta_{ij} = \frac{x_{ij}'}{\sum_{i=1}^{n} x_{ij}'} \tag{7}$$

Next, calculate the information entropy for the j^{th} indicator:

$$H_j = -\frac{1}{lnp} \sum_{j=1}^{p} \eta_{ij} \ln\left(\eta_{ij}\right) \tag{8}$$

Finally, convert to obtain the objective weight:

$$W_j'' = \frac{1 - H_j}{\sum_{j=1}^{p} (1 - H_j)} \tag{9}$$

In this paper, we combine the CRITIC weighting method and the entropy method to obtain the combined weight for the j^{th} indicator:

$$W_j = \frac{W_{j}{}' W_j''}{\sum_{j=1}^{p} W_{j}{}' W_j''} \tag{10}$$

2.2 EWMA Control Chart

The principle of the EWMA (Exponentially Weighted Moving Average) control chart is the exponential weighted moving average method. Moving average calculates the arithmetic mean of the most recent n observations, with the latest observation replacing the earliest one. Weighted moving average improves upon this by assigning equal weights to each observation within a fixed span, with the weights decreasing for previous observations. The EWMA method further refines the weighted moving average and was proposed by Robert in 1959 [6]. It considers the fact that newer observations reflect more effective information and avoids the need to use all historical data, which would require increasing storage space.The construction process of the EWMA control chart is as follows:

Construct the recursive relationship for the exponentially weighted moving average statistic:

$$Z_t = \alpha Z_{t-1} + (1-\alpha)x_t \tag{11}$$

Where Z_t and Z_{t-1} are the weighted moving average values at time t and $t-1$, respectively, Z_0 is usually set as the target value μ_0 when the process is under control, and x_t is the observation at time t. α is the weight, where $0 \leq \alpha \leq 1$. A smaller α gives more influence on the latest observation, making it more sensitive to small fluctuations. By using the recursive relationship, we find that the current sample has the highest weight, and the weights decrease geometrically for previous samples. Specifically, the weight for the first sample is the smallest.

$$Z_t = \alpha^t Z_0 + (1-\alpha)\sum_{i=0}^{t-1} \alpha^i x_{t-i} \tag{12}$$

Group the EWMA statistic values into intervals of size c and plot them to create the control chart. If the standard deviation of the original sample data is denoted as σ, then the standard deviation of the EWMA statistic Z_t is calculated as:

$$\sigma_{Z_t}^2 = \sigma^2\left(\frac{\alpha}{2-\alpha}\right)\left[1-(1-\alpha)^{2t}\right] \tag{13}$$

According to the 3σ rule of statistical process control, the control center line is μ_0, and the upper and lower control limits are:

$$UCL = \mu_0 + 3\sigma_{Z_t}, LCL = \mu_0 - 3\sigma_{Z_t} \tag{14}$$

3. Controlled Simulations

3.1 Experiment Settings

In this paper, 5 simulated controlled experiments were conducted to test the sensitivity of the EWMA control chart based on CRITIC and entropy methods in detecting small drifts using Python programs. Each experiment had a training set and a test set, both with 1000 samples. The weights for dimension reduction were obtained from the training set using CRITIC and entropy methods, and then applied to the test set for dimension reduction. Finally, the control charts for the training and test sets were plotted on the same graph for comparison. To investigate the sensitivity of the control chart construction methods in detecting mean and variance variations, specific settings were made for the 5 experiments, which are listed below. For ease of experimentation without loss of generality, the assumption is that the original sample data in these experiments follows a multivariate normal distribution with 5 indicators, as shown in Table 1.

Table 1: Comparison of training set and test set

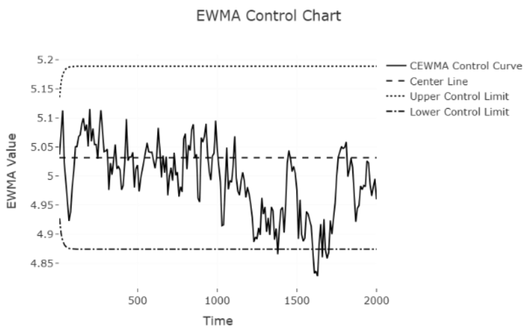
Comparison	Mean	Covariance Matrix	Drift
1	Different	Same, Diagonal Matrices	-0.05
2	Different	Same, Non-Diagonal Matrices	+0.05
3	Same	Different, Diagonal Matrices	+0.5
4	Same	Different, Non-Diagonal Matrices	+0.5
5	Different	Different, Diagonal Matrix for Training Set, Non-Diagonal Matrix for Test Set	

3.2 Experiment Results

Generate 5 groups of training set samples (Sample 1) and test set samples (Sample 2) randomly from a multivariate normal distribution as prescribed data for 5 experiments accordingly.

In Experiment 1, the range of mean vector elements for Sample 1 was $5 \leq \mu_1 \leq 5.05$, while for

Sample 2 it was $4.95 \leq \mu_2 \leq 5$. A slight decrease of 0.05 in the mean was observed in the test set compared to the training set. Both sets had the same covariance matrix, which was diagonal with diagonal elements whose range were $0.5 \leq cov \leq 1$. The control chart in Figure 1 showed that the center line centered around 5.03, exhibited consistent fluctuations in the training set data up to the 1000th sample. However, the test set data on the right side displayed a notable downward trend, crossing below the control lower limit near the 1600th sample. This indicates that despite of the correlation between different indicators, the control chart construction method is highly sensitive to small mean drifts when assuming equal fluctuations.



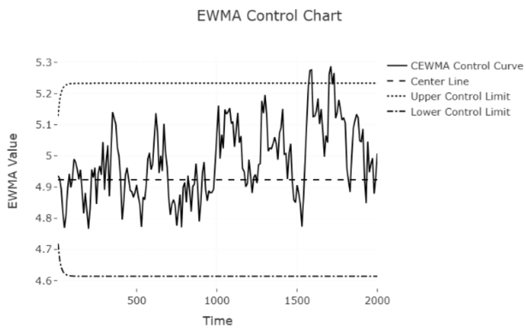
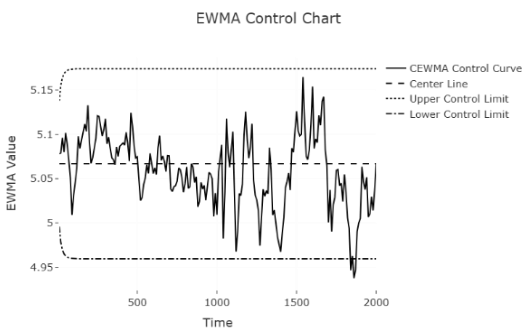


Figure 1: Control chart of Experiment 1   Figure 2: Control chart of Experiment 2

In Experiment 2, the range of mean vector elements for Sample 1 was $4.95 \leq \mu_1 \leq 5$, while for Sample 2 it was $5 \leq \mu_2 \leq 5.05$. A slight increase of 0.05 in the mean was observed in the test set compared to the training set. Both sets had the same non-diagonal covariance matrix, which was diagonal with diagonal elements whose range were $0.5 \leq cov \leq 1$. The control chart in Figure 2 showed that the center line centered around 4.96, exhibited consistent fluctuations in the training set data up to the 1000th sample. However, the test set data displayed a significant upward trend, crossing above the control upper limit near the 1600th and 1700th samples. This suggests that when considering the correlation between different indicators, the control chart construction method presented demonstrates high sensitivity to small mean drifts under the assumption of equal fluctuations.



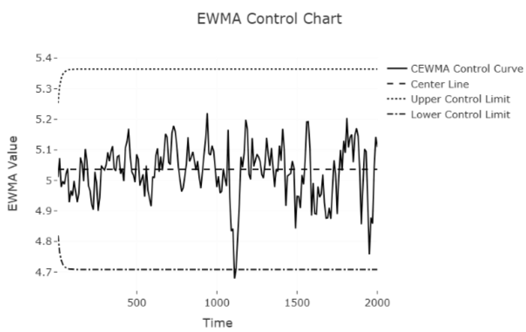


Figure 3: Control chart of Experiment 3   Figure 4: Control chart of Experiment 4

In Experiment 3, both samples had the same mean vector with a range of $5 \leq \mu \leq 5.1$. The covariance matrices of the two samples were different but both diagonal matrices. The range of the diagonal elements of the covariance matrix for sample 1 was $0.1 \leq cov1 \leq 0.3$, while for Sample 2 it was $0.6 \leq cov2 \leq 0.8$. This resulted in a slight amplification of the test set's covariance matrix compared to the training set. The control chart in Figure 3 showed that the center line centered approximately 5.06, exhibited small and uniform fluctuations in the training set data up to the 1000th sample. However, the test set data on the right side displayed a significant increase in fluctuations, frequently approaching and crossing the control upper and lower limits, with a crossing below the control lower limit near the 1800th sample. It shows that when assuming equal mean values, the control chart construction method is highly sensitive to small amplifications in fluctuations despite of the correlation between different indicators.

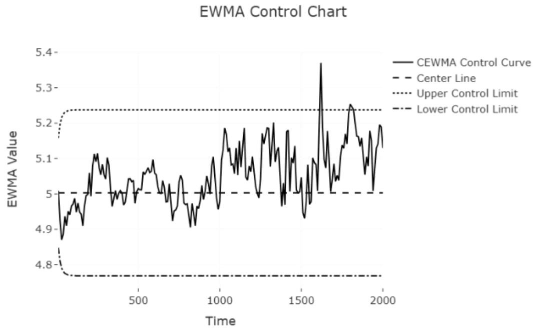


Figure 5: Control chart of Experiment 5

In Experiment 4, both sets had the same mean vector, with a range of $5 \leq \mu \leq 5.1$. The covariance matrices of the two samples were different and non-diagonal. For Sample 1, the range of the elements of the covariance matrix was $0.5 \leq cov1 \leq 0.6$, while for Sample 2 it was $1 \leq cov2 \leq 1.1$. This resulted in a slight amplification of the diagonal elements of the test set's covariance matrix compared to the training set. The control chart in Figure 4 showed that the center line centered approximately 5.06, exhibited small and uniform fluctuations in the training set data up to the 1000th sample. However, the test set data displayed a significant increase in fluctuations, frequently approaching and crossing the control upper and lower limits, with a crossing below the control lower limit near the 1200th sample. It can be concluded that given the correlation between different indicators, this control chart construction method is still highly sensitivity to small amplifications in fluctuations under the assumption of equal mean values.

In Experiment 5, the range of mean vector elements in Sample 1 was $5 \leq \mu_1 \leq 5.05$, while for Sample 2 it was $5.1 \leq \mu_2 \leq 5.15$. This caused a slight upward drift of 0.1 in the test set mean compared to the training set. Both of covariance matrix elements in Sample 1 and 2 had a range of $0.5 \leq cov1 \leq 1$. Whereas, Sample 1 had a diagonal covariance matrix while Sample 2 had a non-diagonal covariance matrix. The control chart in Figure 5 showed that the center line initially centered around 5, exhibited consistent fluctuations in the training set data up to the 1000th sample. Yet the test set data displayed a significant upward trend, causing the center line to drift towards 5.1. Additionally, noticeable amplification in fluctuation magnitude was observed with the test set data crossing above the control upper limit near the 1600th sample. It generates that the control chart construction method is highly sensitive to small mean drifts and amplifications in fluctuations in light of the correlation between different indicators.

4. Conclusions

Considering the complexity of production processes and the growing demand for quality control of multi-indicator variables, this paper proposes a combined weighting method using CRITIC and entropy for dimension reduction in multivariate EWMA control charts. Through 5 sets of simulation experiments, the advantages of this approach turn out as follows: 1) Sensitivity to small variations in mean and volatility; 2) Consideration of correlations between indicators, maximizing information representation; 3) Free of knowledge of variable distributions being allowed, facilitating practical application. However, there is still room for further improvement. Future research can explore: 1) Create additional indicators with superior effectiveness to assess importance of different variables; 2)

Explore non-linear dimension reduction methods incorporating indicator importance and relationships.

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