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Preliminary Analysis of Double Limits

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Abstract: The limit of a univariate function is a critical concept in advanced mathematics, serving as the theoretical foundation for differentiation and integration. When it comes to the limit of bivariate functions, many students have limited exposure to bivariate functions during their high school years. This, coupled with the scarce explanations in textbooks, leads to some misconceptions in understanding the limit of bivariate functions. This paper delves into the concept of the limit of bivariate functions, introduces and solves the limit of bivariate functions by exemplifying the method of determining the limit of univariate functions; It also explains the double limit and iterated limit and distinguishes between them with practical examples, highlighting their connection through a theorem. By preliminarily analyzing the limit of bivariate functions, we hope to inspire math enthusiasts and deepen their understanding of double limits.

1. Introduction

Advanced Mathematics is an essential foundational course for non-mathematical disciplines in science and engineering. In actual teaching practices in application-oriented private undergraduate colleges, one of the challenging points for students is the methodology of limit calculation. The concept of limits is closely related to continuity, differentiation, integration, and series in advanced mathematics, thus serving as the foundation of advanced mathematics as a whole [1]. Specifically, the calculation methods for univariate function limits are relatively well-established. However, when transitioning from the concept of univariate function limits to bivariate function limits, almost all versions of advanced mathematics textbooks treat the concept, calculation method, and understanding of bivariate function limits superficially, without any in-depth discussion. To facilitate students' learning and mastery of the relevant content of bivariate function limits, this paper, based on many years of actual teaching of advanced mathematics and teaching practice and experience related to bivariate function limits, elaborates on the concepts, calculation methods [2], and the connection and difference between double limits and iterated limits. We hope to provide some inspiration for enthusiasts of advanced mathematics and deepen their understanding and knowledge of double limits.

2. Concept of Bivariate Function Limits

Suppose that a bivariate function $f(x,y)$ is defined within a punctured neighborhood $\mathring{U}(P_0)$

of point $P_0(x_0, y_0)$, and A is a constant. If for any given $\varepsilon > 0$, there always exists $\delta > 0$, such that for all points $P(x, y) \in \overset{\circ}{U}(P_0)$ satisfying the inequality of $0 < |PP_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, $|f(x, y) - A| < \varepsilon$ always holds, then A is called the limit of the function $f(x, y)$ when $(x, y) \to (x_0, y_0)$, which is denoted as $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = A$ or $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = A$.

Based on this definition, the following two points should be noted:

First, like the definition of univariate function limits, bivariate functions only need to be defined within a certain punctured neighborhood $\overset{\circ}{U}(P_0)$ of $P_0(x_0, y_0)$. The existence of a limit is irrelevant to whether the function is defined at this point [1]. That is, the definition of a function and its limit are two completely different concepts. The former is a relatively static concept, while the latter is a concept of dynamic tendency.

Example 1: solve $\lim\limits_{\substack{x \to 0 \\ y \to 2}} \dfrac{\tan(xy)}{x}$.

Solution: $\because \lim\limits_{\substack{x \to 0 \\ y \to 2}} \dfrac{\tan(xy)}{x} = \lim\limits_{\substack{x \to 0 \\ y \to 2}} \dfrac{\tan(xy)}{xy} \cdot y = \lim\limits_{\substack{x \to 0 \\ y \to 2}} \dfrac{\tan(xy)}{xy} \cdot \lim\limits_{y \to 2} y = 2$

but the function $f(x, y) = \dfrac{\tan(xy)}{x}$ is undefined at the point $(0, 2)$, but the limit exists.

Second, the arbitrariness of variable trajectories in the limit process of bivariate functions. The way the moving point $P(x, y)$ approaches point $P_0(x_0, y_0)$ in the definition of a bivariate function limit is arbitrary. It can be a linear or a curvilinear approximation to P_0, which is completely different from the univariate function limit where the point can only move left or right along the number axis. Under the condition of the existence of the limit, no matter how complicated the way the moving point $P(x, y)$ approaches point $P_0(x_0, y_0)$, each way tends to the final limit value A [2]. In specific teaching, the contrapositive of this feature is often used to illustrate that the required bivariate function limit does not exist. That is, as the moving point $P(x, y)$ approaches point $P_0(x_0, y_0)$ in different ways, the bivariate function tends to different values, thus the limit of this bivariate function does not exist.

Example 2: Prove $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Proof: Assume that the moving point $P(x, y)$ approaches the origin $(0, 0)$ along the line $y = kx$, then:

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - k^2x^2}{x^2 + k^2x^2} = \frac{1 - k^2}{1 + k^2}$$

From this, it can be seen that as the moving point $P(x, y)$ approaches the origin $(0, 0)$ along different lines, due to different values of k, the function tends to different values, so the limit of this bivariate function does not exist.

3. Using Methods Similar to Univariate Function Limits to Solve Double Limits

Double limits and univariate function limits share many similarities from definition to properties, thus we can employ methods similar to univariate function limits to solve double limits [3-4], such as the Squeeze Theorem, equivalent infinitesimals, variable substitution, cancelling zero factors, important limits, and L'Hôpital's Rule [5], etc. Let's give a simple example to illustrate.

Example 3: Solve $\lim\limits_{\substack{x\to\infty\\y\to\infty}} \frac{x+y}{x^2+y^2}$.

Solution: When $xy \neq 0$, there is $0 \le \left|\frac{x+y}{x^2+y^2}\right| \le \frac{|x|+|y|}{x^2+y^2} \le \frac{|x|+|y|}{2|xy|} = \frac{1}{2|y|}+\frac{1}{2|x|} \to 0 (x\to\infty, y\to\infty)$

So $\lim\limits_{\substack{x\to\infty\\y\to\infty}} \frac{x+y}{x^2+y^2} = 0$

Example 4: Solve $\lim\limits_{\substack{x\to 0\\y\to 0}} \frac{\sin(x^2+y^2)}{\ln(1+x^2+y^2)}$.

Solution: The fraction in the formula shows a homogeneous structure "x^2+y^2", and when $(x,y)\to(0,0)$, $x^2+y^2\to 0$, we can set $x^2+y^2=u$, then we have $\lim\limits_{\substack{x\to 0\\y\to 0}} \frac{\sin(x^2+y^2)}{\ln(1+x^2+y^2)} = \lim\limits_{u\to 0}\frac{\sin u}{\ln(1+u)} = \lim\limits_{u\to 0}\frac{u}{u} = 0$.

Example 5: Solve $\lim\limits_{\substack{x\to 0^+\\y\to 0^+}} (1+xy)^{\frac{1}{\sin xy}}$.

Solution: Since the original formula is a binary power exponential function [6], so $(1+xy)^{\frac{1}{\sin xy}} = \exp(\frac{1}{\sin(xy)}\ln(1+xy))$, $\frac{1}{\sin(xy)}\ln(1+xy) = \frac{(xy)}{\sin(xy)}\ln(1+xy)^{\frac{1}{xy}}$, then

$\lim\limits_{\substack{x\to 0^+\\y\to 0^+}} \frac{(xy)}{\sin(xy)} = 1$, $\lim\limits_{\substack{x\to 0^+\\y\to 0^+}} \ln(1+xy)^{\frac{1}{xy}} = \ln \lim\limits_{\substack{x\to 0^+\\y\to 0^+}} (1+xy)^{\frac{1}{xy}} = \ln e = 1$, So we have $\lim\limits_{\substack{x\to 0^+\\y\to 0^+}} (1+xy)^{\frac{1}{\sin xy}} = e$

Example 6: Solve $\lim\limits_{(x,y)\to(0,0)} \frac{\exp(1/x^2+y^2)}{\ln(x^2+y^2)}$.

Solution: This question belongs to the $\frac{\infty}{\infty}$ type indeterminate limit [7]. By applying L'Hôpital's Rule for multivariate functions [5], we get that the original formula equals

$$\lim_{(x,y)\to(0,0)} \frac{\exp(1/(x^2+y^2))\cdot(-2x\cdot x-2y\cdot y)/(x^2+y^2)^2}{(2x\cdot x+2y\cdot y)/(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} -\frac{\exp(1/(x^2+y^2))}{x^2+y^2} = -\infty.$$

4. Differences and Some Connections between Double Limits and Iterated Limits

In the system of calculus textbooks for bivariate functions, the limit of a bivariate function is sometimes also referred to as a double limit, and in the integral calculus of bivariate functions, the double integral of a bivariate function is also called an iterated integral. Hence, some students in their learning process erroneously extrapolate that a double limit could also be referred to as an iterated limit [8-10]. These are completely different concepts, and we'll explain them in detail through their definitions and examples.

When calculating the limit of a bivariate function $f(x,y)$, if the moving point (x,y) approaches point (x_0,y_0) in any manner, then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ is referred to as a double limit, and there's only one limit symbol in the formula.

If the moving point (x,y) approaches point (x_0,y_0) following a piecewise linear path like $(x,y)\to(x,y_0)\to(x_0,y_0)$ or $(x,y)\to(x_0,y)\to(x_0,y_0)$, then the limits $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y)$ and $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$ are referred to as iterated limits. There are two limit symbols in the formula, indicating two limit processes, specifically two univariate function limits, which is also known as an iterated limit.

Example 7: Discuss the double limit and iterated limit of the bivariate function $f(x,y)=\dfrac{xy}{x^2+y^2}$ at point $(0,0)$.

Solution: Suppose the moving point $P(x,y)$ approaches the origin $(0,0)$ along the line $y=kx$, then: $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{xy}{x^2+y^2}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{kx^2}{x^2+k^2x^2}=\dfrac{k}{1+k^2}$

From this, it can be seen that as the moving point $P(x,y)$ approaches the origin $(0,0)$ along different lines, due to different values of k, the function tends to different values, so the limit of this bivariate function does not exist.

However $x\neq 0, y\to 0$ $\lim\limits_{y\to 0}\dfrac{xy}{x^2+y^2}=0$, then $\lim\limits_{x\to 0}\lim\limits_{y\to 0}\dfrac{xy}{x^2+y^2}=0$, similarly there is $\lim\limits_{y\to 0}\lim\limits_{x\to 0}\dfrac{xy}{x^2+y^2}=0$

From this example, it can be seen that the double limit of the function does not exist, but both iterated limits exist and are equal. This shows that the existence and equality of two iterated limits cannot guarantee the existence of the double limit.

Example 8: Discuss the double limit and iterated limit of the bivariate function $f(x,y)=x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x}$ at $(0,0)$.

Solution: Because $0<\left|x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x}\right|\leq|x|+|y|\leq 2\sqrt{x^2+y^2}$

From the Squeeze Theorem, we have: $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=0$

But when $y\neq 0, x\to 0$, $\sin\dfrac{1}{x}$ does not exist, that is, $\lim\limits_{x\to 0}(x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x})$ does not exist, so the iterated limit $\lim\limits_{y\to 0}\lim\limits_{x\to 0}(x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x})$ does not exist, and similarly, $\lim\limits_{x\to 0}\lim\limits_{y\to 0}(x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x})$ also does not exist.

From this example, it can be seen that the double limit of the function exists, but both iterated limits do not exist. This shows that the existence of a double limit cannot guarantee the existence of iterated limits, and it certainly cannot guarantee the equality of the two iterated limits.

Example 9: Prove that $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^3+y^2}{x^2+y^2}$ does not exist.

Proof 1: Referring to the method in Example 2, assume that the moving point $P(x,y)$

approaches the origin $(0,0)$ along the line $y = kx$, then: $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^3+y^2}{x^2+y^2}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^3+k^2x^2}{x^2+k^2x^2}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x+k^2}{1+k^2}=\frac{k^2}{1+k^2}$

From this, it can be seen that as the moving point $P(x,y)$ approaches the origin $(0,0)$ along different lines, due to different values of k, the function tends to different values, so the limit of this bivariate function does not exist.

Proof 2: Let's first calculate the two iterated limits:

$\lim\limits_{x\to 0}\lim\limits_{y\to 0}\frac{x^3+y^2}{x^2+y^2}=\lim\limits_{x\to 0}\frac{x^3}{x^2}=\lim\limits_{x\to 0}x=0$, and $\lim\limits_{y\to 0}\lim\limits_{x\to 0}\frac{x^3+y^2}{x^2+y^2}=\lim\limits_{y\to 0}\frac{y^2}{y^2}=1$.

Because the two iterated limits are not equal, $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^3+y^2}{x^2+y^2}$ does not exist.

From this example, we can see that if the two iterated limits of the function exist but are not equal, then the double limit must not exist (this conclusion is often used as a method to determine the nonexistence of double limits).

So, is there an intrinsic connection between double limits and iterated limits [11]? The answer is yes, under certain specific conditions, there are some connections between the two. Let me introduce you to the following theorem:

Theorem: Suppose that the double limit of function $f(x,y)$ exists at (x_0,y_0), i.e., $\lim\limits_{(x,y)\to(x_0,y_0)}f(x,y)=A$, and suppose there is $\lim\limits_{x\to x_0}f(x,y)=\varphi(y)$, where $\varphi(y)$ is a function of y, then we have $\lim\limits_{y\to y_0}(\lim\limits_{x\to x_0}f(x,y))=A$.

Proof: From condition $\lim\limits_{(x,y)\to(x_0,y_0)}f(x,y)=A$, we know that for any given $\varepsilon>0$, there always exists $\delta_1>0$. When $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta_1$,

i.e., $0<|x-x_0|<\delta_1$, $0<|y-y_0|<\delta_1$, we have $|f(x,y)-A|<\frac{\varepsilon}{2}$;

Also, $\lim\limits_{x\to x_0}f(x,y)=\varphi(y)$ exists, i.e., when y is in some neighborhood $\mathring{U}(y_0;\delta_2)$ of y_0, for any given $\varepsilon>0$, there always exists $\delta_1>0$ such that when $0<|x-x_0|<\delta_1$, we have

$|f(x,y)-\varphi(y)|=|\varphi(y)-f(x,y)|<\frac{\varepsilon}{2}$;

Set $\delta=\min\{\delta_1,\delta_2\}$, When $0<|x-x_0|<\delta$, $0<|y-y_0|<\delta$, We have

$|\varphi(y)-A|=|\varphi(y)-f(x,y)+f(x,y)-A|\le|f(x,y)-\varphi(y)|+|f(x,y)-A|<\varepsilon$

Then we have $\lim\limits_{y\to y_0}\varphi(y)=A$, that is $\lim\limits_{y\to y_0}(\lim\limits_{x\to x_0}f(x,y))=A$ holds, hence proved.

Similarly, we can get the following conclusion:

Assume that the double limit of function $f(x,y)$ exists at (x_0,y_0), that is, $\lim\limits_{(x,y)\to(x_0,y_0)}f(x,y)=A$, and there is $\lim\limits_{y\to x_0}f(x,y)=\varphi(x)$, where $\varphi(x)$ is a function of x, then we have $\lim\limits_{x\to x_0}(\lim\limits_{y\to y_0}f(x,y))=A$.

By combining the above two conclusions, we can quickly get the following three propositions [9]:

a. If the double limit and both iterated limits exist, then they must be equal.

b. If the double limit and one of the iterated limits exist, then they must be equal.

c. If both iterated limits exist but are not equal, then the double limit must not exist.

5. Summary

Through the above superficial analysis of double limits, it is hoped that readers can deeply understand and comprehend the concept of limits of bivariate functions. At the same time, this plays an important role in exercising and improving the flexibility and divergent level of thinking. Hopefully, this article can serve as a reference and improvement for learning the basic concepts related to higher mathematics.

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