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Model Misspecification in Portfolio Optimization

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Abstract: This paper investigates the situation in Merton (1969) model that volatility is a constant rather than a stochastic process, then points out that this is a model misspecification since it doesn't match the real market. Next, the HJB equation with stochastic volatility is derived through stochastic control, thereby calibrate model misspecification.

1. Introduction

Nowadays, in modern control theory, one essential problem is how to deal with stochastic control problem. In order to satisfy stochastic control problem, a state equation must always be maximized/minimized within a set of admissible controls with respect to a certain value function. One of the main approaches to solving such problems is dynamic programming principle (DPP). It was proposed by Bellman [1], DPP effectively deals with the above optimization problems under discrete control, and provides a theory basis for the Hamilton-Jacobi-Bellman (HJB) equation in the context of continuous control. For a function, by providing both necessary and sufficient conditions for optimal control, the HJB equation is typically formulated as a nonlinear PDE (partial differential equation) in the value function, which is also the solution to the equation (value function) [2]. When solution is obtained, optimal control can be achieved by maximizing the Hamiltonian, which involved in the HJB equation [3].

The desirability of stochastic control has led to its application being extended to the financial field: as investment banks and asset management companies seek to efficiently build investment portfolios in the process of economic globalization, this has become a crucial aspect of their operations. Merton's research on continuous time portfolio optimization began in 1969 [4]. He introduced a realistic re-balancing policy and made the assumption which stock price follow a GBM model. Through the use of DPP and the HJB equation, Merton was able to derive a closed-form solution for the CRRA (constant relative risk aversion) utility function, which enabled to determine the optimal control and maximum return.

With the in-depth study of scholars, they found that one of the biggest problems of Merton model is that the constant volatility hypothesis cannot capture a large amount of market information, which leads to the corresponding deviation of pricing. However, using the assumptions of the Heston model [5], it is possible to describe real market movements more naturally. In this type of model, both the stock price and the volatility are stochastic processes. Li [6] constructed a problem framework for simulating investment based on Heston model, and obtained the explicit solution of optimal control by solving a set of ODE (ordinary differential equation) under the power utility function. It is very instructive for investment activities in the real market and can help investors find the optimal return. But the drawback is the stock process and volatility have no correlation, they just obey the same Brownian motion.

To our knowledge, currently, there is no reference specifically exploring how to calibrate Merton (1969) model misspecification and derive the correct HJB equation. This paper gives a general method for deriving HJB equation and optimal control through DPP and Taylor expansion under stochastic volatility model.

2. DPP and HJB equation

There are four main methods to solve stochastic control problems: maximum principle, martingale, DPP, machine learning. One of the most classical mathematical methods is DPP, which Bellman first proposed it in 1957. Subsequently, many scholars have proved and developed on the basis of above, e.g. Nisio [7] and Yong, etc.

Typically, we want some restrictions on the control process in order to optimize. For example, a $\mathcal{F}$- progressively measurable process π is required to be adapted to state process X_t. The definition of admissible control is a control process that satisfies some constraints, where $\mathcal{A}$ representations the set of all admissible controls Therefore, π is called optimal control and $\pi \in \mathcal{A}$.

For simplicity, given a bounded and continuous function ϕ where the range is $\phi: \mathbb{R} \to \mathbb{R}$, then the performance criterion is:

$$V^{\pi}(x) = \mathbb{E}[\phi(X_T^{\pi})] \tag{1}$$

The value function uses expectation to predict future returns. Here we do not consider discount factor and infinite time. Under the range $[0, T] \times \mathbb{R} \to \mathbb{R}$, define value function V:

$$V(x) = \sup_{\pi \in \mathcal{A}_{[0,T]}} V^{\pi}(x) = \sup_{\pi \in \mathcal{A}_{[0,T]}} \mathbb{E}[\phi(X_T^{\pi})] \tag{2}$$

Assume $s \leq t \in [0, T]$, DPP can be written as:

$$V(s, X_s) = \sup_{\pi \in \mathcal{A}_{[s,t]}} \mathbb{E}[V(t, X_t^{\pi}) \mid \mathcal{F}_s] \tag{3}$$

We assume state process X_t satisfies the following SDE (stochastic differential equation) with drift $a(t, X_t, \pi_t)$ and volatility $b(t, X_t, \pi_t)$, where $W_t \sim \mathcal{N}(0,1)$ is Brownian motion; and σ_t for the same reason:

$$dX_t^{\pi} = a(t, X_t, \pi_t)dt + b(t, X_t, \pi_t)dW_t \tag{4}$$

$$d\sigma_t = c(t, X_t, \pi_t)dt + d(t, X_t, \pi_t)dZ_t \tag{5}$$

$$dW_t dZ_t = \rho dt \tag{6}$$

where correlation $-1 < \rho < 1$. We need to maximize:

$$V(t, X) = \sup_{\pi \in \mathcal{A}} J(t, X, \pi) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[\Phi(X_T^{\pi}) + \int_0^T f(t, X_t, \pi_t)dt\right] \tag{7}$$

where J is objective function of the control problem, Φ is bequest function [2], f is instantaneous utility function.

In Merton's model (1969), under constant volatility, his HJB equation is:

$$0 = \sup_{\pi \in \mathcal{A}} \left\{f(t, X_t, \pi_t) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial X}a + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}b^2\right\} \tag{8}$$

Due to the limitations of the times and technological development, Merton (1969) didn't realize the model misspecification at that time. Since volatility is stochastic process, in the process of DPP, we need to consider the influence of σ_t, and there should be three variables t, σ, x in the value function; Afterwards, use Taylor expansion and Ito's lemma to obtain HJB equation. It is worth noting that the HJB equation obtained by above method for $V(t, X)$ and $V(t, X, \sigma)$ are different, which directly affects the expression of optimal control.

If the conclusion of equation (8) is directly applied to the case where the volatility is a stochastic process, it will make a mistake. In order to calibrate the model misspecification, we should derive HJB equation correctly. The method is: use a discrete time process firstly to discretize equation (7), then expand it to a continuous time process by taking $\Delta t \to 0$. The approximation process is:

$$J(t, X, \pi) = \mathbb{E}[\Phi(X_T) + \textstyle\sum_{t=1}^{T} f(t, X_t, \pi_t)\Delta t] \tag{9}$$

The initial value X_0 and σ_0 is constant, which is the constraints of equation (9). After using DPP, value function V becomes:

$$V(t, X_t, \sigma_t) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[V(t+1, X_{t+1}, \sigma_{t+1}) + f(t, X_t, \pi_t)\Delta t] \tag{10}$$

Through Taylor expansion and Ito's lemma for value function at time $(t+1)$:

$$\begin{aligned} V(t+1, X_{t+1}, \sigma_{t+1}) \quad &= V(t, X_t, \sigma_t) + \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial X}\Delta X_t + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(\Delta X_t)^2 \\ &+ \frac{\partial V}{\partial \sigma}\Delta\sigma + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}(\Delta\sigma_t)^2 + \frac{\partial^2 V}{\partial\sigma\,\partial V}\rho\Delta\sigma\Delta X + o(\Delta t) \end{aligned} \tag{11}$$

Substitute (10) into (11), it can obtain:

$$\begin{aligned} V(t+1, X_{t+1}, \sigma_{t+1}) \quad &= \frac{\partial V}{\partial X}(a\Delta t + b\Delta W_t) + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(a\Delta t + b\Delta W_t)^2 + \frac{\partial V}{\partial t}\Delta t + o(\Delta t) + V(t, X_t, \sigma_t) \\ &+ \frac{\partial V}{\partial \sigma}(c\Delta t + d\Delta Z_t) + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}(c\Delta t + d\Delta Z_t)^2 + \frac{\partial^2 V}{\partial\sigma\,\partial V}\rho(a\Delta t + b\Delta W_t)(c\Delta t + d\Delta Z_t) \end{aligned} \tag{12}$$

Unite like terms and simplify:

$$\begin{aligned} 0 = \sup_{\pi \in \mathcal{A}} \quad &\Big\{\mathbb{E}\Big[\frac{\partial V}{\partial X}(a\Delta t + b\Delta W_t) + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(a\Delta t + b\Delta W_t)^2 + \frac{\partial V}{\partial t}\Delta t + f(t, X_t, \pi_t)\Delta t \\ &+ \frac{\partial V}{\partial \sigma}(c\Delta t + d\Delta Z_t) + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}(c\Delta t + d\Delta Z_t)^2 \\ &+ \frac{\partial^2 V}{\partial\sigma\,\partial V}\rho(a\Delta t + b\Delta W_t)(c\Delta t + d\Delta Z_t) + o(\Delta t)\Big]\Big\} \end{aligned} \tag{13}$$

Since V has differentiable and smooth properties, take $\Delta t \to 0$ under continuous form. Use tower property, quadratic variation and independence property, the HJB equation becomes:

$$0 = \sup_{\pi \in \mathcal{A}} \Big\{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial X}a + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}b^2 + \frac{\partial V}{\partial \sigma}c + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}d^2 + \frac{\partial^2 V}{\partial\sigma\,\partial X}\rho bd + f(t, X_t, \pi_t)\Big\} \tag{14}$$

with terminal condition:

$$V(T, X) = \Phi(X_T) \tag{15}$$

About equation (14), for every backward dt from time 0, instantaneous utility function f increment is accumulated for the objective function J (although its limit is 0); In addition, it also affects the time state $\frac{\partial V}{\partial t}$, state process $\frac{\partial V}{\partial X}$ and $\frac{\partial^2 V}{\partial X^2}$, volatility state $\frac{\partial V}{\partial \sigma}$ and $\frac{\partial^2 V}{\partial \sigma^2}$, joint state $\frac{\partial^2 V}{\partial\sigma\partial X}$ at dt moment. The impact of this moment on the state will continue until the next moment, until terminal time T. As time goes by under optimal control, the value function of HJB equation, e.g. the corresponding PDE, remains unchanged since the left hand side of the equation is always 0. This indicates from another perspective that the optimal control should be a dynamic expression rather than a fixed value under constant volatility, thus equation (14) calibrate model misspecification [8].

To this extent, the derivation of the HJB equation after calibrating model misspecification has been completed. For Merton (1969), the optimal control under model misspecification is:

$$\pi^*_{misspecification} = -\frac{a\frac{\partial V}{\partial X}}{b^2 X\frac{\partial^2 V}{\partial X^2}} \tag{16}$$

The optimal control under calibration is:

$$\pi^*_{calibration} = -\frac{a\frac{\partial V}{\partial X}+c\frac{\partial V}{\partial \sigma}}{b^2 X\frac{\partial^2 V}{\partial X^2}+d^2 X\frac{\partial^2 V}{\partial \sigma^2}+2\rho bd\frac{\partial^2 V}{\partial \sigma\, \partial V}} \tag{17}$$

In fact, the total solving process is divided into five steps, which are summarized as follows:

(1) Determine state process X_t;

(2) Calculate HJB equation correctly (depend on constant volatility or stochastic volatility);

(3) Take partial derivative of $\frac{\partial V}{\partial \pi}$ and set it equal to 0 to get expression of π;

(4) Make a suitable ansatz for value function V, it is usually based on structure and experience. For relatively simple optimal control problems, it can be solved explicit solution by mathematical method. However, if the form is more complex, it may not be able to find explicit solution, thus it can only be simulated by numerical method.

(5) Substitute π into step (4) and combine with terminal condition to obtain the explicit solution of value function V.

3. Conclusion

HJB equation is one of the classical methods for solving stochastic control problems, it is widely used in the field of financial mathematics. When we face various stochastic control problems, it is necessary to consider the characteristics of different stochastic processes to get the corresponding HJB equation. In this paper, we calibrate Merton (1969) model misspecification, use DPP and Taylor expansion to derive a general method for solving HJB equation and optimal control in the case of stochastic volatility.

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