Advances in Computer, Signals and Systems (2023)
Clausius Scientific Press, Canada

DOI: 10.23977/acss.2023.070603
ISSN 2371-8838 Vol. 7 Num. 6

Global Dynamics of Predator-Prey Model with Special Holling IV Functional Response

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Keywords: Predator-prey model, Holling IV functional response, slow-fast system, global stability, limit cycle

Abstract: In this paper, we will focus on the number, type, local stability, and global stability of the positive equilibrium of the predator-prey model with special Holling IV functional response. When the positive equilibrium is a weak focus, it can be of order one and stable, of order one and unstable, of higher order. Moreover, if the equilibrium is unique and unstable, there exists a limit cycle surrounding it.

1. Introduction

The predator-prey model describes the dynamic relationship between predator and prey and plays a significant role in explaining the inherent laws of ecosystems and describing and predicting changes in ecosystems. This model is often used in mineral prediction to study the internal laws of mineral formation. The predator-prey model was proposed by Lotka and Volterra. The model was improved by Holling, and he proposed three response functions of Holling type I, II, and III [1, 2]. Later Andrews proposed an inhibition function based on experiments [3], which is known as Holling type IV response function [4].

The typical Holling I, II, and III types represent three situations where the predator quantity increases uniformly with the prey, the predator rate decreases when reaching saturation, and the situation when it is difficult to capture when the prey density is low. The Holling IV type is an inhibition function, for the predator decreases as the quantity of prey increases when it reach a certain value.

Our study focus on Holling type IV functional responses. Recent research on predator-prey models with this type of response, such as Zhang, Wu and Zhou [5], Ren and Li [6], Shang, Qiao, Duan and Miao [7], Zhao and Shen [8], etc., focus on latest predator-prey models, and those models mostly metamorphosis from the classical models to different degrees.

This paper will discuss the classical predator-prey model with Holling IV functional response. There are also some research results about it, such as Ruan and Xiao[9] proved that the system exists numerous bifurcations, including saddle-node bifurcation, supercritical and subcritical Hopf bifurcation, homoclinic bifurcation; Li and Zhu[10] proved that for the systems with Holling III and Holling IV type functional responses the cyclicity of the limit periodic set is at most 2 through the singular perturbation theory, and indicate the regions in parameter space where the corresponding limit periodic set has cyclicity at most one or two.

We will mainly explore the dynamics of the system from the geometric perspective of the critical curve, discuss the number, type, local and global stability of the equilibria of the system, and the existence of the limit cycle in the system.

2. Predatory-prey models and the basic properties

The classic predatory-predatory model is as follows[11]:

$$\begin{aligned} \dot{x} &= rx\left(1-\frac{x}{K}\right) - f(x)y, \\ \dot{y} &= y(-\alpha + \beta f(x)), \end{aligned} \tag{1}$$

In (1), x represents the prey, y represents the predator, r is the intrinsic growth rate of the prey, K is the environmental capacity, α is the mortality rate of the predator, β is the transformation efficiency, and f is the functional response, in other words, the predation rate of predators changes with the density of prey. Here we require $x, y, r, K, \alpha, \beta$ are positive.

The simplified Holling Type IV functional response function is as follows:

$$f(x) = \frac{x}{ax^2+b} \tag{2}$$

The simplified Holling IV functional response has fewer parameter variables than the original one.

To study the dynamics of the differential equation (1) with the functional response (2), the parameters of the equation are first reduced equivalently.

According to relevant research methods of various predator-prey models[12], we transform the parameters r and K into 1 through the scale transformation of space and time. Then we suppose $c = \frac{b}{a}, \tilde{f} = af$ and readjust the scale of prey by supposing $\tilde{y} = \frac{y}{a}$. Thus system (1) becomes the following differential equation:

$$\begin{aligned} \dot{x} &= x(1-x) - \tilde{f}(x)\tilde{y}, \\ \dot{y} &= \alpha\tilde{y}(-1 + \gamma\tilde{f}(x)), \end{aligned} \tag{3}$$

here $\gamma = \frac{\beta}{\alpha a}$. Because we do not restrict the symbols of a, we require $\gamma \neq 0$, $y \neq 0$ and $x \in (0,1)$. The response function becomes:

$$\tilde{f}(x) = \frac{x}{x^2+c} \tag{4}$$

This action inverts f into a single-parameter function where $c \neq 0$. Note that the new variable needs $\tilde{y}\gamma > 0$ because the symbols of both are consistent to a.

When $a < 0$, the sign change of the variable affects the Jacobian matrix of the differential equation and thus the calculation of the type and other properties of the equilibria, but has no effect on the number of the equilibria, and the position will only be axisymmetric along the x-axis.

2.1. The number of the equilibrium

For the number of singularities, we give the following proposition:

Proposition 1. For the system (3), when its functional response is (4), the following conclusion

holds:
(i) If $c \in [1, +\infty)$:
- When $\gamma \in (1 + c, +\infty)$, the system has only one equilibrium;
- When $\gamma \in (-\infty, 0) \cup (0, 1 + c]$, the system has no equilibrium.

(ii) If $c \in (0,1)$:
- When $\gamma \in (2\sqrt{c}, 1 + c)$, the system has two equilibria;
- When $\gamma \in \{2\sqrt{c}\} \cup [1 + c, +\infty)$ the system has one equilibrium;
- When $\gamma \in (-\infty, 0) \cup (0, 2\sqrt{c})$, the system has no equilibrium.

(iii) If $c \in (-1,0)$:
- When $\gamma \in (-\infty, 0) \cup (0, 1 + c)$, the system has one equilibrium;
- When $\gamma \in [1 + c, +\infty)$, the system has no equilibrium.

(iv) If $c \in (-\infty, -1]$:
- When $\gamma \in (-\infty, 1 + c)$, the system has one equilibrium;
- When $\gamma \in [1 + c, 0) \cup (0, +\infty)$, the system has no equilibrium.

Proof. Looking at the second term of the differential equation, it is easy to see that the condition of $-1 + \gamma f(x) = 0$ guarantees the existence of the equilibrium and we can verify that $\tilde{y}\gamma > 0$ holds at the same time. Then we just need to check the image of the function f and find the number of the intersections of the curves and the horizontal line $y = \frac{1}{\gamma}$.

2.2. The distribution of the equilibrium

In this part, we will discuss mainly about distribution and also some other properties of the equilibrium. Suppose $\tau = \alpha t$, the system (3) is transformed to:

$$\begin{aligned} \alpha x' &= x(1 - x) - f(x)y, \\ y' &= y(-1 + \gamma f(x)), \end{aligned} \tag{5}$$

Notice that (3) and (5) has the same global dynamics. When $0 < \alpha \ll 1$, τ is called slow time, and the system (5) is slow system. Correspondingly, the system (3) is called the fast system. To explore the global dynamics of the system (3) and (5), we consider the singular perturbation theory[see e.g. 13, 14, 15, 16].

When $\alpha \to 0$, we get the layer system:

$$\begin{aligned} \dot{x} &= x(1 - x) - f(x)y, \\ \dot{y} &= 0, \end{aligned} \tag{6}$$

and the reduced system:

$$\begin{aligned} 0 &= x(1 - x) - f(x)y, \\ y' &= y\big(-1 + \gamma f(x)\big), \end{aligned} \tag{7}$$

notice the critical set:

$$M_0 = \{(x, ay) \in \mathbb{R}_+^2 | x(1 - x) - f(x)y = 0\} \tag{8}$$

it consists of the positive or negative part of the axis-y and the critical curve:

$$S_0 = \{y = C(x) \triangleq (1 - x)(x^2 + \quad c) | x \in (0,1)\} \tag{9}$$

then transform the reduced system (7) to 1-dimension system:

$$C'(x)\frac{dx}{d\tau} = C(x)\big(-1 + \gamma f(x)\big) \tag{10}$$

and we obtain the follow proposition.

Proposition 2. For critical curves S_0 and reduced systems (7), the following conclusions hold:

(a) When $c \in [1, +\infty)$, the curve S_0 strictly decrease in $(0,1)$, and the unique equilibrium (if exists) of the system(7) is stable and hyperbolic;

(b) When $c \in \left(\frac{1}{3}, 1\right)$, the curve S_0 strictly decrease in (0,1), and if there are two equilibria, the left one is stable and hyperbolic, and the right one is unstable and hyperbolic; if there is exactly one equilibrium, then it is stable and hyperbolic;

(c) When $c = \frac{1}{3}$, S_0 monotonically decreases in (0,1) and has a unique critical point at $x = x_c = \frac{1}{3}$. If there are two equilibria, the left one is stable and hyperbolic and the other one is unstable and hyperbolic; and if there is a unique equilibrium, it is also stable;

(d) When $c \in \left(0, \frac{1}{3}\right)$, the curve S_0 has a local minimum point x_1 and a local maximum point x_2, and the curve decreases in $(0, x_1), (x_2, 1)$ and increases in (x_1, x_2). Assume that the reduced system has two equilibria, then both x_L and x_R locate in $(x_1, 1)$, and x_L is stable(unstable) and hyperbolic if it locates on the right-(left-)hand side of x_2, x_R is stable(unstable) and hyperbolic if it locates on the left-(right-)hand side of x_2. If there is a unique equilibrium, then the equilibrium is stable and hyperbolic when it locates in $(0, x_1) \cup (x_2, 1)$, and it is unstable and hyperbolic when it locate in (x_1, x_2);

(e) When $c \in (-1,0)$, the curve S_0 has a local maximum point x_1 such that $0 < \sqrt{-c} < x_1 < 1$. Assume the reduced system has a unique equilibrium x_*, then it satisfies $x_* \neq \sqrt{-c}$, and it is stable(unstable) and hyperbolic when it is on the left-(right-) hand side of x_1.

(f) When $c \in (-\infty, -1]$, the curve S_0 strictly increase in $(0,1)$, and if the system has a unique equilibrium, it is stable and hyperbolic.

Proof. First we get the differentiation of $C(x)$:

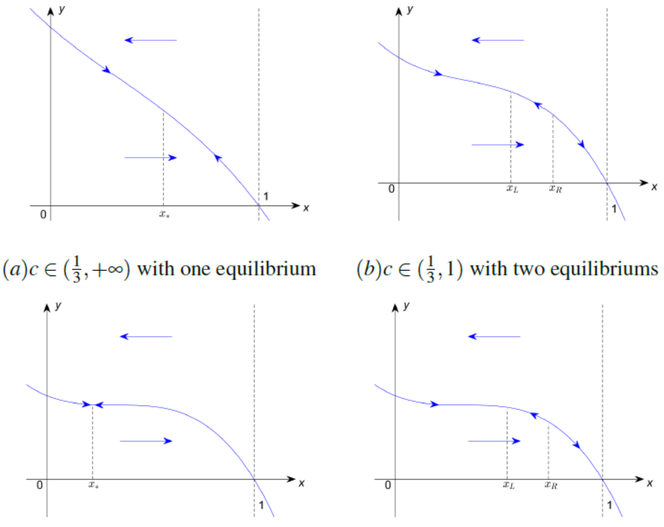
$$C'(x) = -3x^2 + 2x - c \tag{11}$$

It reaches the maximum value $y_c = \frac{1}{3} - c$ at $x_c = \frac{1}{3}$, and the boundary values at this time are $C'(0) = -c$ and $C'(1) = -c - 1$. Next we study the layer system (6), we can see that when $f > 0$, the upper part of the curve would flow horizontally to the left, while the bottom of the curve would flow horizontally to the right, and the opposite would be true for $f < 0$. By computing the sign of $\dot{y}$ and $\dot{x}$, we get the dynamics of the system (7) restrict on S_0.

The Figure 1, Figure 2, Figure 3, and Figure 4 completely describe the proposition 2 from (a) to (f).

Notice the case when $c \in \left(0, \frac{1}{3}\right)$. If the system has two equilibria, by computing $C'(c)$ and $C'\left(\sqrt{c}\right)$, we have $0 < x_1 < c < x_2 < 1$ and $\sqrt{c} < x_2$ ($\sqrt{c} > x_2$) when $0 < c < \frac{1}{4}$ ($\frac{1}{4} < c < \frac{1}{3}$). So picture (d_{11}), (d_{12}) only exist when $0 < c < \frac{1}{4}$ and (d_{14}), (d_{15}) only exist when $\frac{1}{4} < c < \frac{1}{3}$. If there is one equilibrium, we have $0 < x_* < c < x_2$ or $x_* = \sqrt{c}$. Picture (d_{21}), (d_{22}), (d_{23}) display the former case and (d_{23}), (d_{24}), (d_{25}) display the latter case when $0 < c < \frac{1}{4}, c = \frac{1}{4}, \frac{1}{4} < c < \frac{1}{3}$. This conclusion will be useful in the discussion later.

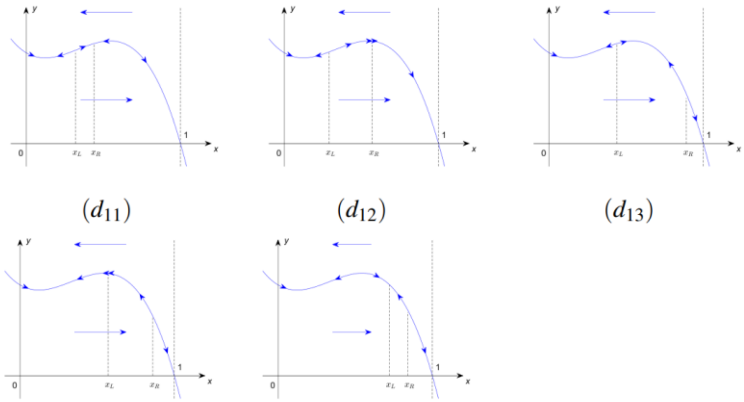
Returning to the system (3) and (5), notice that equilibria do not depend on whether α is non-zero or not. Therefore, the position of the equilibria is the same to Proposition 2. According to Fenichel's theorem[13], if the equilibrium of the Proposition 2 is hyperbolic, then the corresponding equilibrium of the system (3) and (5) have the same hyperbolic properties. In addition, local stability can be observed combining with the orbits of the system (6) and (7). If both are stable, then we find the equilibria of the system (3) and (5) are also locally stable. We will accurately discuss the stability of the equilibria in the next part.



$(a) c \in (\frac{1}{3}, +\infty)$ with one equilibrium $(b) c \in (\frac{1}{3}, 1)$ with two equilibriums

$(c_1) c = \frac{1}{3}$ with one equilibrium $(c_2) c = \frac{1}{3}$ with two equilibriums

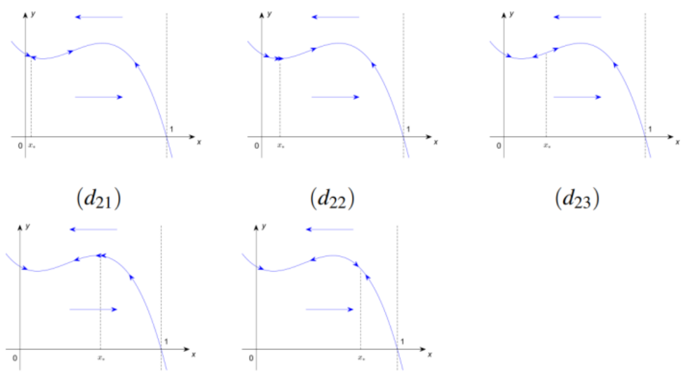
Figure 1: The dynamics of system (7) with corresponding parameters



(d_{11}) (d_{12}) (d_{13})

(d_{14}) (d_{15})

Figure 2: The dynamics of system (7) with c $\in \left(0, \frac{1}{3}\right)$ and two equilibria



(d_{21}) (d_{22}) (d_{23})

(d_{24}) (d_{25})

Figure 3: The dynamics of system (7) with c $\in \left(0, \frac{1}{3}\right)$ and one equilibrium

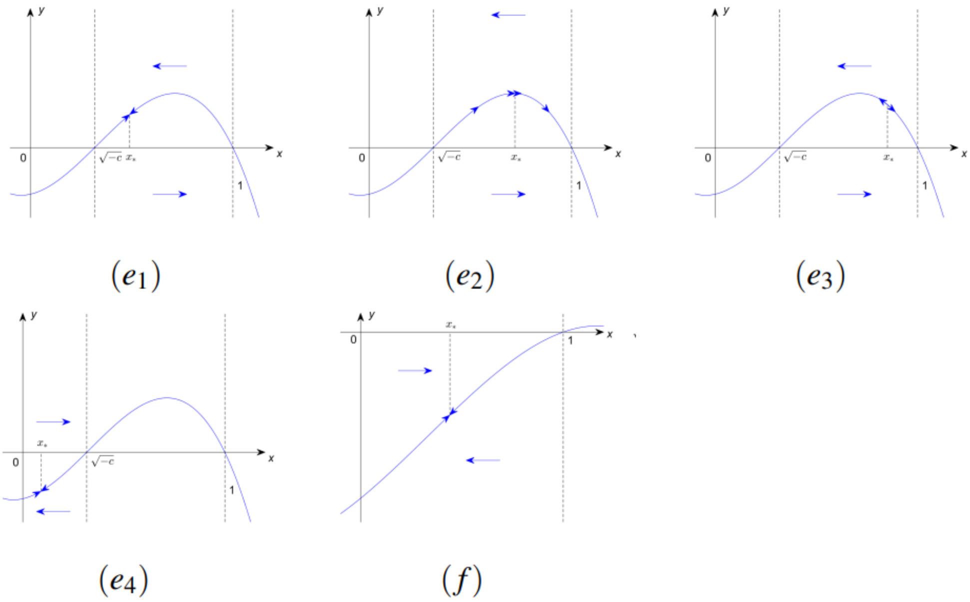


Figure 4: The dynamics of system (7) with corresponding parameters

2.3 The local and global dynamics

In order to avoid the influence of the sign of a on the Jacobian matrix and determinant, we return to the original differential equation system (1). We still transform the r and K into 1 with the same method. If $a > 0$, we do the same process as $\tilde{f}(x) = af(x)$ and $\tilde{y} = \frac{y}{a}$, else the process will be $\tilde{f}(x) = -af(x)$ and $\tilde{y} = -\frac{y}{a}$. Then we have $\gamma, y, f(x) > 0$, and the system will be transformed into the system (3) with the following functional response:

$$\tilde{f}(x) = \text{sign}(a) \cdot \frac{x}{x^2+c} \tag{12}$$

and the proposition 1, 2 still hold.

3. The type and local stability of the equilibrium

First we compute the Jacobian matrix of the system (3) with functional response (12):

$$J(x,y) = \begin{pmatrix} 1-2x-f'(x)y & -f(x) \\ \alpha\gamma f'(x)y & -\alpha\big(1-\gamma f(x)\big) \end{pmatrix} \tag{13}$$

and we compute the Jacobian matrix of the boundary equilibrium $(0,0)$:

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -a \end{pmatrix} \tag{14}$$

when $c \neq -1$, the system (3) has another boundary equilibrium $(1,0)$, and the Jacobian matrix:

$$J(1,0) = \begin{pmatrix} -1 & -\text{sign}(a) \cdot \frac{1}{1+c} \\ 0 & -\alpha + \text{sign}(a) \cdot \frac{\alpha\gamma}{1+c} \end{pmatrix} \tag{15}$$

For $\alpha > 0$ we know $(0,0)$ is a saddle, and the type of $(1,0)$ is determined by γ and c. For equilibrium locates in $\{(x,y) \in \mathbb{R}_+^2 | x \in (0,1)\}$, we have $y = \frac{x(1-x)}{f(x)}$ and $-\alpha\big(1-\gamma f(x)\big) = 0$, then we compute the characteristic function:

$$\mathcal{F}_J = \lambda^2 - C'(x)f(x)\lambda + \alpha\gamma x(1-x)f'(x) \tag{16}$$

here we redefine $C(x) \triangleq \mathrm{sign}(a) \cdot (1-x)(x^2+c)$ and thus $C'(x) = \mathrm{sign}(a) \cdot (-3x^2+2x-c)$.

If there are two equilibria E_L and E_R, by proposition 1, we have $c \in (0,1)$ and $\gamma \in \left(2\sqrt{c}, 1+c\right)$, thus the equilibrium $(1,0)$ is a stable node. Consider the characteristic function (16), we know E_R is a saddle and E_L is a stable (unstable) node or focus when $C'(x_L) < 0$ ($C'(x_L) > 0$), and is a weak focus or center when $C'(x_L) = 0$, and we will discuss it separately. Thus, we obtain a conclusion:

(1) For $c \in (0,1)$, E_R is a saddle;

(2) For $c \in [\frac{1}{3}, 1)$, E_L is a stable focus or node;

(3) For $c \in \left(\frac{1}{4}, \frac{1}{3}\right)$, E_L is a stable focus or node, an unstable focus or node, a weak focus or center according to its position;

(4) For $c \in \left(0, \frac{1}{4}\right)$, E_L is an unstable focus or node;

If the system(3) has a unique equilibrium E_*. By using the same method, we obtain the following discussion:

(1) For $c \in [1, +\infty)$, E_* is a stable focus or node;

(2) For $c \in \left(\frac{1}{3}, 1\right)$, E_* is a stable focus or node when $x_* \neq \sqrt{c}$ and it is stable and degenerate when $x_* = \sqrt{c}$;

(3) For $c = \frac{1}{3}$, E_* is a weak focus or center when $x_* = \frac{1}{3}$, and it is the same as (1) and (2) for the other cases;

(4) For $c \in \left(0, \frac{1}{3}\right)$: When $x_* \neq \sqrt{c}$, E_* is a stable focus or node if $x_* \in (0, x_1)$ and unstable focus or node if $x_* \in (x_1, x_2)$, it is a weak focus or center when $x_* = x_1$; when $x_* = \sqrt{c}$, E_* is an unstable degenerate node if $x_* \in (x_1, x_2)$ and a stable degenerate node if $x_* \in (x_2, 1)$, and it is nil-potent when $x_* = x_2 = \sqrt{c} = \frac{1}{2}$;

(5) For $c \in (-1,0)$: if $x_* \in \left(\sqrt{-c}, 1\right)$, E_* is a saddle; if $x_* \in \left(0, \sqrt{-c}\right)$, E_* is a stable focus or node;

(6) For $c \in (-\infty, -1]$, we have E_* is a stable focus or node.

Thus we have provided the types and local stability of equilibrium in most cases.

Next, we will discuss in detail the case when the equilibrium is a weak focus or center. According to related book[17], for such a system:

$$\begin{aligned} \dot{x} &= -\sigma y + p(x,y), \\ \dot{y} &= \sigma x + q(x,y), \end{aligned} \tag{17}$$

with $\sigma > 0$ and

$$\begin{aligned} p(x,y) &= \textstyle\sum_{i+j\geq 2} a_{ij}\, x^i y^j, \\ q(x,y) &= \textstyle\sum_{i+j\geq 2} b_{ij}\, x^i y^j, \end{aligned} \tag{18}$$

if the following equation is non-zero, then the first order Lyapunov constant of the system is:

$$\mathcal{L}_1 = \frac{3\pi}{2}\sigma^{-1}\Big(3(a_{30}+b_{03}) + (a_{12}+b_{21}) - \sigma^{-1}\big(2(a_{20}b_{20} - a_{02}b_{02}) - a_{11}(a_{20}+a_{02}) + b_{11}(b_{20}+b_{02})\big)\Big) \tag{19}$$

and the equilibrium is a weak focus of order 1, which is unstable(stable) when $\mathcal{L}_1 > 0$ ($\mathcal{L}_1 < 0$).

Consider the methods from relevant research[18], suppose $u = x - x_*, v = y - y_*$, and change the time scale by supposing $u = f(x_*)\epsilon$ and $v = \omega\eta \triangleq \sqrt{\alpha\gamma f(x_*)f'(x_*)y_*}\eta$, then the system is transformed into:

$$\begin{aligned}\frac{d\epsilon}{d\tau} &= -\eta - \frac{(2+f''(x_*)y_*)f(x_*)}{2\omega}\epsilon^2 - f'(x_*)\epsilon\eta \\ &\quad - \frac{f'''(x_*)y_*f^2(x_*)}{6\omega}\epsilon^3 - \frac{1}{2}f''(x_*)f(x_*)\epsilon^2\eta + O(|(\epsilon,\eta)|^4), \\ \frac{d\eta}{d\tau} &= \epsilon + \frac{\alpha\gamma f''(x_*)y_*f^2(x_*)}{2\omega^2}\epsilon^2 + \frac{\alpha\gamma f'(x_*)f(x_*)}{\omega}\epsilon\eta \\ &\quad + \frac{\alpha\gamma f'''(x_*)y_*f^3(x_*)}{6\omega^2}\epsilon^3 + \frac{\alpha\gamma f''(x_*)f^2(x_*)}{2\omega}\epsilon^2\eta + O(|(\epsilon,\eta)|^4),\end{aligned} \tag{20}$$

then we compute the first order Lyapunov constant:

$$\mathcal{L}_1 = \frac{3\pi f(x_*)}{4\omega f'(x_*)}\mathcal{L}_0 \tag{21}$$

here $\mathcal{L}_0 = -f'''(x_*)y_*f(x_*)f'(x_*) + (2 + f''(x_*)y_*)(f''(x_*)f(x_*) + f'(x_*)^2)$ and we have $C'(x_*) = -3x_*^2 + 2x_* - c = 0$, thus:

$$\mathcal{L}_0 = \frac{16(1-x_*)x_*^5(24x_*^2-19x_*+3)}{(x_*^2+c)^6} \tag{22}$$

the sign of $\mathcal{L}_1$ depends on $z(x_*) = 24x_*^2 - 19_*^2 + 3$ and $z(x)$ has two real roots:

$$r_1 = \frac{19-\sqrt{73}}{48}, \quad r_2 = \frac{19+\sqrt{73}}{48} \tag{23}$$

and we can discuss for the case when the equilibrium is a weak focus or center:

(1) For $c \in \left(\frac{1}{4}, \frac{1}{3}\right)$ and the system (3) has two equilibria, if E_L is a weak focus or center, then $x_L = x_2 \in \left(\frac{1}{3}, \frac{1}{2}\right) \subset (r_1, r_2)$ which means E_L is a stable weak focus of order 1;

(2) For $c \in \left(0, \frac{1}{3}\right)$ and the system (3) has a unique equilibrium, if E_* is a weak focus or center, then $x_L = x_1 \in \left(0, \frac{1}{3}\right)$. Thus when $x_* = x_1 < r_1$, E_* is an unstable weak focus of order 1; when $x_* = x_1 > r_1$, E_* is a stable weak focus of order 1; when $x_* = x_1 = r_1$, $\mathcal{L}_1 = 0$ is not the first order Lyapunov constant and should calculate a higher order differentiation of the Poincaré map, and we won't further discuss it in this paper;

(3) For $c = \frac{1}{3}$ and $\gamma = \frac{4}{3}$, we have $x_* = x_c = \frac{1}{3} \in (r_1, r_2)$, so E_* is a stable weak focus of order 1.

In this part we obtain the type and local stability of the equilibrium in almost all the cases. Next, we will discuss the global dynamics of the system (3).

3.1 Global stability of the stable equilibrium

For the equilibrium that is locally stable, we focus on whether it is globally stable or not. We have the following proposition:

Proposition 3. For system (3) with functional response (12) and $c \notin (0, \frac{1}{3})$, if the equilibrium is locally stable, then it is globally stable.

Proof. Notice that when $c \notin (0, \frac{1}{3})$, the system has a unique equilibrium if exists. Then consider the Dulac criterion [19]. Set

$$H(x, y) = \frac{1}{f(x)y} = -\mathrm{sign}(a) \cdot \frac{x^2+c}{xy} \tag{24}$$

and correspondingly we have

$$\big(P(x,y),Q(x,y)\big)=\Big(x(1-x)f(x)y,\alpha y\big(-1+\gamma f(x)\big)\Big) \tag{25}$$

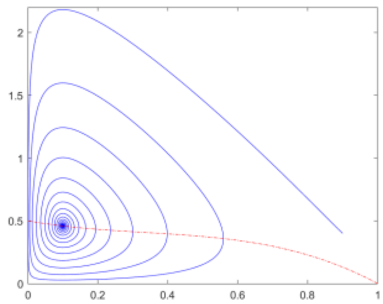
then:

$$\Delta:=\frac{\partial(HP)}{\partial x}+\frac{\partial(HQ)}{\partial y}=\frac{C\prime(x)}{y}=\begin{cases}<0, & x\in(0,1), & c\in\left(\frac{1}{3},+\infty\right),\\ \leq 0, & x\in(0,1), & c=\frac{1}{3},\\ <0, & x\in\left(0,\sqrt{-c}\right), & c\in(-1,0),\\ <0, & x\in(0,1), & c\in(-\infty,-1],\end{cases} \tag{26}$$

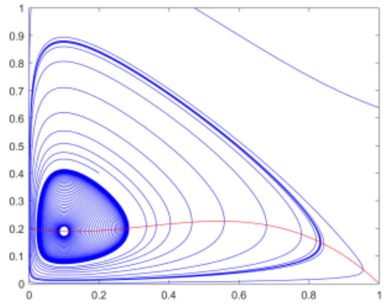
here for the case when $\Delta\leq 0$, the equality holds if and only if $x=\frac{1}{3}$, but $x=\frac{1}{3}$ is not an orbit of the system (3) with functional response (12).

According to the Dulac Criterion, we prove that the system does not have a cycle, so the locally stable equilibrium is also globally stable.

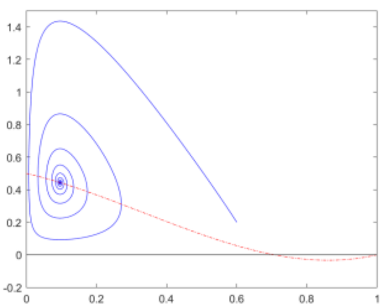
In this way, we partially obtained the relationship between global stability and local stability of the system. However, locally stable equilibrium may not necessarily be globally stable when $c\in\left(0,\frac{1}{3}\right)$, as we will demonstrate below. As shown in Figure 5, we show the orbit of system (3) with a unique equilibrium in four different parameters, where all equilibrium are stable.



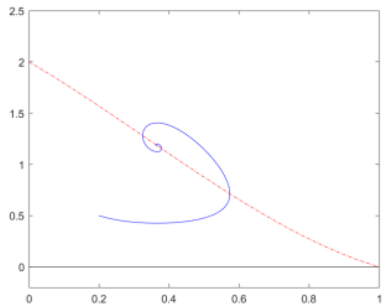
$(i)c=0.5,\gamma=5.1$



$(ii)c=0.2,\gamma=2.1$



$(iii)c=-0.5,\gamma=5.1$



$(iv)c=-2,\gamma=5.1$

Figure 5: The orbit of system (3) with corresponding parameters

In Figure 5, orbits in figures(i), (iii), (iv) are consistent with the result of the proposition, that is, there is no limit cycle within the corresponding range. However, the situation shown in the figure(ii)is different. In this case, the system (3) has two limit cycles, of which the outer limit cycle is semi-stable, it is the ω-limit of its inner orbits and the α-limit of its outer orbits; the inner limit cycle is an unstable limit cycle; there is a stable equilibrium inside the inner limit cycle.

Here we discuss the global dynamics of stable equilibrium, next we will discuss the global dynamics of the unstable equilibrium.

3.2 Global dynamics of the unstable equilibrium

For the equilibrium that is not stable, we also focus on the limit cycle. We have the following proposition:

Proposition 4. For system (3) with functional response (12), if the equilibrium is unique locally unstable, then there exists at least a limit cycle surrounding it.

Proof. According to the previous conclusion, system (3) satisfies the propositional condition only if $c \in \left(0, \frac{1}{3}\right)$ and $x_* \in [x_1, x_2)$. Note that the x-axis and y-axis are invariant sets in the system, the boundary equilibrium $(0,0)$ is a saddle and $(1,0)$ is a saddle or a stable and degenerate node; when $x = 1$ and $y > 0$, we have $\dot{x} < 0$. Thus, the region

$$\{(x,y)|x \in (0,1), y \in (0,+\infty)\} \tag{27}$$

is positive invariant.

Next refer to methods in relevant research[20], give the upper bound of the region $y = k - \alpha\gamma x$, and we have:

$$\alpha\gamma\dot{x} + \dot{y} = \alpha\gamma(1-x)x - \alpha y \tag{28}$$

notice that for $x \in (0,1)$, we have $y \geq k - \alpha\gamma$ and $x(x-1) < 1$, thus:

$$\alpha\gamma\dot{x} + \dot{y} \leq \alpha\gamma - \alpha(k - \alpha\gamma) = \alpha\gamma(1+\alpha) - \alpha k \tag{29}$$

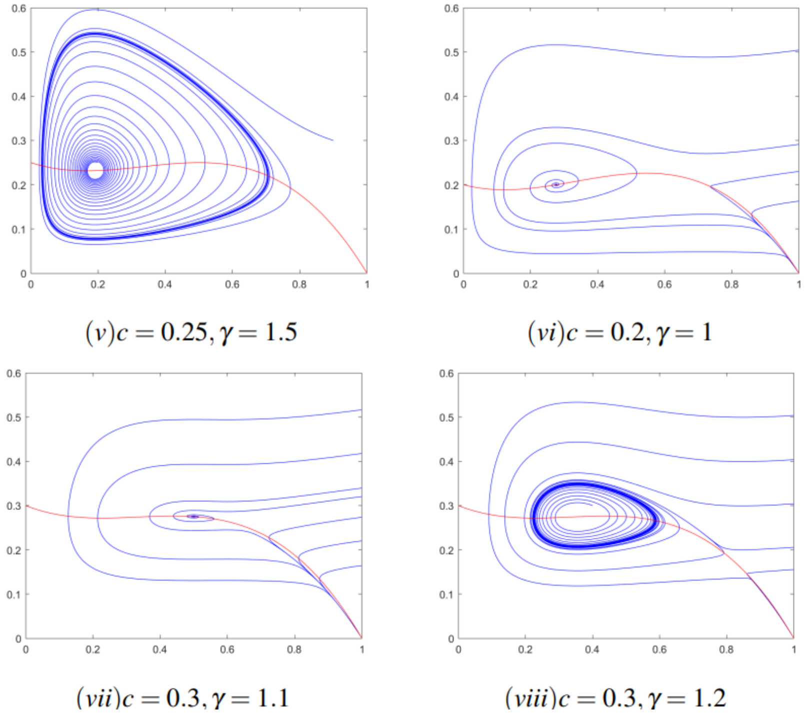
when $k > \gamma(1+\alpha)$, the orbit through $y = k - \alpha\gamma x$ all positively enter the region:

$$\Omega_0 = \{(x,y)|x \in (0,1), y \in (0, k - \alpha\gamma x)\} \tag{30}$$

Thus the region Ω_0 is positive invariant. When k is big enough, the region covers the unstable equilibrium. By removing a neighborhood of the equilibrium, we find an annular field with all the orbit positively enter the field. According to Poincaré-Bendixson annulus theorem, there is at least one limit cycle in the field. Notice that when system has two equilibria, the left one can also be locally unstable. In this case $\gamma < 1 + c$, so the boundary equilibrium $(1,0)$ is a stable node. Thus system (3) has orbits flow to that point, which means the region Ω_0 is not positive invariant. And we will show that in Figure 6.

In Figure 6, (v) is consistent with the result of the proposition with a unique equilibrium that is unstable and has a limit cycle surrounding it. And (vi) is the case we have mentioned where the left equilibrium is unstable and the boundary equilibrium $(1,0)$ is the ω-limit of the orbits. Notice that not all the orbits must flow to $(1,0)$. In Figure 6 (vii), the left equilibrium is stable, and some orbits flow to E_L and others flow to $(1,0)$. Another situation is the case of (viii), there is a stable limit cycle in the system surrounding an unstable equilibrium, and some orbits flow to the limit cycle and others flow to $(1,0)$. We can also compute the equilibria at infinity of the system by Poincaré compactification [see e.g. 21].

In conclusion, when the system (3) has two equilibria and the left one is unstable, the existence of the limit cycle depends on the parameters.



$(v) c = 0.25, \gamma = 1.5$ $(vi) c = 0.2, \gamma = 1$

$(vii) c = 0.3, \gamma = 1.1$ $(viii) c = 0.3, \gamma = 1.2$

Figure 6: The orbit of system (3) with corresponding parameters

4. Conclusion

For system (3) with functional response (12), we prove that the following statements hold:

(1) System (3) has at most two positive equilibria and could have 0, 1, 2 positive equilibrium for different parameters.

(2) When the system has two positive equilibria, the left one could be a stable focus or node, an unstable focus or node, a stable weak focus of order 1 for different parameters and the right one is a saddle.

(3) When the system has a unique positive equilibrium, then:

- The equilibrium could be a stable focus or node, an unstable focus or node,a stable weak focus of order 1, an unstable weak focus of order 1, a weak focus of higher order, a saddle for different parameters.

- The locally stable equilibrium is not always globally stable, but if c $\notin (0,\frac{1}{3})$, the local stability is equivalent to global stability.

- If the equilibrium is locally unstable, then there exists a limit cycle surrounding it.

Acknowledgments

This study was supported by the Program of Sichuan Mineral Resources Research Center (Grant no. SCKCZY2023-ZC006).

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