Curriculum and Teaching Methodology (2023)
Clausius Scientific Press, Canada

DOI: 10.23977/curtm.2023.061204
ISSN 2616-2261 Vol. 6 Num. 12

Application of Euler Method in Discrete Dynamic Systems

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Keywords: Discrete power system, Euler Method, Lorenz system, Holling-Tanner system

Abstract: At first, the paper introduces the research background of dynamic systems and discrete dynamic systems, and then expounds the basic theoretical knowledge of Euler Method and discrete dynamic systems. Based on these theoretical knowledge, we illustrate the application of Euler Method in discrete dynamic systems on autonomous and non-autonomous.In the autonomous discrete system, we expound the famous Lorenz system, and we introduce the two-dimensional Holling-Tanner systemin the non-autonomous discrete system. Finally, by using MATLAB software, we obtain the corresponding results and figure with Euler Method.

1. Introduction

Dynamical system is a rule that describes the change of system variables over time.Originally, it only referred to mechanical systems described by differential equations derived from Newton's classical mechanical theory, which was mainly obtained by Lyapunov and Poincare at the end of the 19th century. Then Nemystskii, Stepanov, Coddington and Levinson elaborated and disposed the properties of dynamical systems which were defined by differential equations. Subsequently, the concept is widely applied in many different branches of science. The theory of modern dynamical systems mainly originates from the work of Kolmogorov,Smale and Anosov.Taking time as the classification standard, it can be divided into continuous dynamic system and discrete dynamic system;Using dimension as the standard, it can be divided into finite dimensional dynamical system and infinite dimensional dynamical system. Considering the state and the dependence between the states, it can be divided into linear dynamic system and nonlinear dynamic system. Therefore, the ordinary differential equation can be regarded as a continuous dynamic system, and the corresponding difference equation can be taken as a discrete dynamic system [1].

At the same time, dynamical system is also closely related to numerical analysis because the results of qualitative analysis can be used as the research premise of numerical analysis. Of course, these data also help to improve the calculation method in numerical analysis, and the results provide materials and methods for the qualitative theoretical research of dynamical system. However, being based on numerical analysis, Euler Method will certainly become a powerful research tool.

2. Basic theoretical knowledge

2.1 Basic Knowledge of Euler Method

2.1.1 Method of Euler

The method of Euler is the simplest numerical solution. The so-called numerical solution is a discretization method, which can be used to approximate the value $y_1, y_2, \cdots, y_n$of an unknown function $y(x)$ at a series of discrete points $x_1, x_2,\cdots, x_n$.Among them, $x_1, x_2,\cdots, x_n$ is given in advance.So,it's called a node;$h = x_{n+1} - x_n (h > 0, n = 0,1,2, \cdots)$ named as size (variable),which it's an equal distance usually.That is to say $x_n = x_0 + ih\,(i = 1,2,3, \cdots)$, $y_1, y_2, \cdots, y_n$ is a numerical solution of the initial value problem [2].

Considering the following initial value problem by using the method of Euler

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \tag{1}$$

We can conclude $y' = f(x_0, y_0)$. Supposing h is equal distance step, $x_1 = x_0 + h$,we get approximately

$$\frac{y(x_1) - y(x_0)}{h} = f(x_0, y_0) \tag{2}$$

When h is sufficiently small. So $y_1 = y_0 + h\, f(x_0, y_0)$.Similarly, $y_2 = y_1 + hf(x_1, y_1)$ when $y(x_1)$= $y(x_0 + 2h)$,⋯; $x_{n+1} = x_0 + (n + 1)h$, we get approximately

$$\frac{y(x_{n+1}) - y(x_n)}{h} = y'(x_n) = f(x_n, y_n) \tag{3}$$

When h is sufficiently small.

In general, we use y_n to represent an approximation of $y(x_n)$, then

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{4}$$

When $x_{n+1} = x_0 + (n + 1)h$

This is the explicit Euler Method, or Euler scheme for short.

Generally, Taylor expansion can be used as a tool to analyze and calculate the accuracy of formulas. To simplify the analysis, we assume that y_n is accurate, that is, the error $y(x_{n+1}) - y_{n+1}$ is estimated on the premise of $y_n = y(x_n)$.This error is called the local truncation error.We have $f(x_n, y_n) = f\big(x_n, y(x_n)\big)$=$y'(x_n)$ from (3). Therefore, the local truncation error is as follows under the Euler scheme:

$$y(x_{n+1}) - y_{n+1} = \frac{h^2}{2} y''(\xi) \approx \frac{h^2}{2} y''(x_n) \tag{5}$$

2.1.2 The implicit Euler format

Since difference is an approximate calculation of differentiation, one of the basic ways to achieve discretization is to directly substitute the difference quotient for the derivative.The derivative $y'(x_{n+1})$ is approximately replaced by the backward difference quotient $\frac{y(x_{n+1}) - y(x_n)}{h}$, we get

$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_{n+1})$$

That is

$$y(x_{n+1}) \approx y_n + hf\big(x_n, y(x_{n+1})\big) \tag{6}$$

It is the implicit Euler format[3].

The same way, according to the local truncation error of Euler scheme, we get

$$y(x_{n+1}) - y_{n+1} \approx -\frac{h^2}{2} y''(x_n) \tag{7}$$

2.2 The discrete dynamical system

Definition 1 Dynamic system

A dynamical system is a semigroup G acting on the space M.That is, there is a mapping

$$T: G \times M \to M$$

$$(g, x) \to T_g(x)$$

that makes

$$T_g \cdot T_h = T_{g \cdot h} \tag{8}$$

It is a discrete dynamical system when G=N or G=Z, and it is a continuous dynamical system when $G = R^+$or $G = R$. G is said to be an invertible dynamical system if it is a group[4].

For example, assuming $f: I \to I$(I represents interval),

$$T^n = f^n = f \cdot f^{n-1} = f \cdot \cdots \cdot f(n), G = N. \tag{9}$$

We can conclude that T^n is a discrete dynamical system.This dynamical system is reversible when f is invertible and generalize the definition to $n \epsilon Z$ in the usual way.

Definition 2 Autonomous and non-autonomous systems

In mathematics, a dynamical system is autonomous if it can be represented by a system of ordinary differential equations whose expressions are irrelevant of the independent variables of the dynamical system. On the contrary, if they are related which can say the system is non-autonomous. Autonomous systems often do not contain time t explicitly in dynamics, but non-autonomous systems often have explicit time t.

Definition 3 Compression mapping

Let X be the metric space, and T be the mapping from X to X, if $\forall x, y \in X$ when $\exists \mu, 0 < \mu < 1$,then

$$d(Tx, Ty) \le \mu d(x, y) \tag{10}$$

So we call T a compression mapping [5].

Theorem 1 the theorem of Compression mapping

Given that X is a complete metric space and T is a compressed map on X, then T has one and only one fixed point. That is, the equation $Tx = x$ has one and only one solution.

3. Application of Euler Method in discrete dynamic system

3.1 Discrete Examples of differential Equations (autonomous)

We consider the ordinary differential equation of Lorenz system [6]:

$$\begin{cases} x' = a(y - x) \\ y' = bx - y - xz \\ z' = cz + xy \end{cases} \tag{11}$$

A chaotic state appears when the system parameter is -1∈ $[-1.59,7.75]$. The nonlinear terms of this equation are only A and B, and there is no obvious time variable, so the system is autonomous. Jules Henri Poincaré, a French mathematician, once pointed that there was a discrete system that productively accompanied non-autonomous systems. His idea is reflected in (11) of Lorenz model that we solve the plane α when (11) of each rail line cuts across α,and then we can define the mapping $T\colon \alpha \to \alpha.$ Then,we argues that $T(X) = Y$ is considered to be true when the track of (11) that goes through point X passes through plane α again and it intersects the plane at point Y.T must be smooth if it exists, and a fixed point Z is on T,so a track that passes through Z must return to Z while it forms a closed loop by Theorem 1.As shown in figure 1 that plane α is replaced by an arc l, we can obtain the mapping $T\colon l \to l$ if (11) is a planar system.

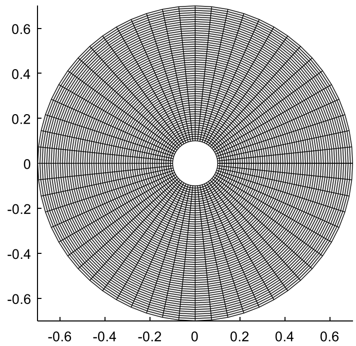

Figure 1: Planar graph

Therefore, the problem of periodic solutions that it is derived from a group of differential equationsis, transformed into the fixed point problem of discrete dynamic systems. The fixed point of a discrete dynamic system can be realized by a computer program. The closed orbit of the system of differential equations may exhibit a very complex "macarone" shape when T is "chaotic", and the orbit could go though repeatedly the plane α at the points of a cantor set.

Using Euler Method (4) to discretize it, we can get

$$\begin{cases} x_n = x_{n-1} + t\big(a(y_{n-1} - x_{n-1})\big) \\ y_n = y_{n-1} + t(bx_{n-1} - x_{n-1}z_{n-1} - y_{n-1}) \\ z_n = z_{n-1} + t(x_{n-1}y_{n-1} + cz_{n-1}) \end{cases} \tag{12}$$

Among them, $a, b, and\ c$ are constants, and t is the time step.This is a differential equation, its mapping form as follows:

$$\begin{cases} x \to x + ta(y - x) \\ y \to y + t(bx - xz - y) \\ z \to z + t(xy + cz) \end{cases} \tag{13}$$

The following is implemented for the Lorenz model system track diagram drawing program.

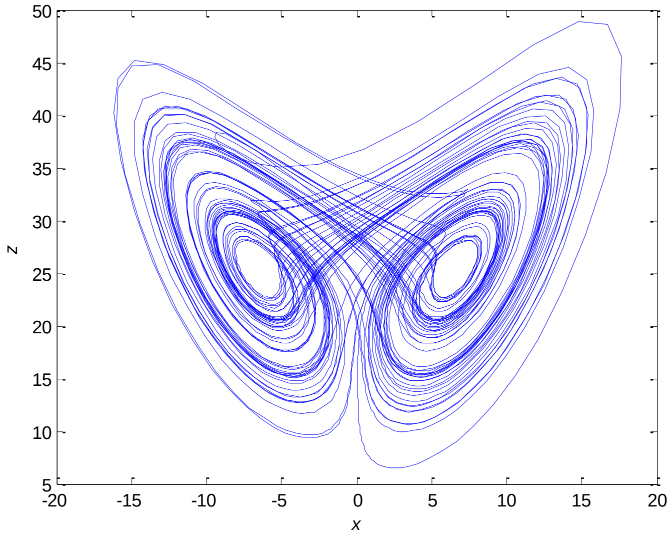

Figure 2: Tracks of Lorenz system

A "singular attractor" can be seen in Figure 2, but it has not been fully analyzed theoretically so far.

3.2 Discretization Examples of differential Equations (non-autonomous)

Here, we consider the Euler discretization of the two-dimensional Holling -Tanner system.The theroy of Holling-Tanner has always been an important topic in population ecology, which has been paid close attention by many researchers in the field of biological mathematics and ecology. Holling-Tanner system is a complex discrete dynamical system. Over the years, Holling - Tanner ecological model updated to improve constantly. Such as reaction functionof Holling-I, Holling-II, Holling-III, Holling-IV, and the half - scale dependence of this reaction function on the Holling-Tanner system. In the following study, we will consider the Euler discretization of two-dimensional non-autonomous Holling-Tanner systems.

The two-dimensional non-autonomous Holling-Tanner system is of the form [7] as follows:

$$\begin{cases} \frac{dN}{dt} = N\left[r\left(1-\frac{N}{K}\right)-\frac{kP}{N+D}\right] \\ \frac{dP}{dt} = P\left[s\left(1-\frac{hP}{N}\right)\right] \end{cases} \tag{14}$$

We use the Euler Method (3.1.4) to discretize the concrete process:

①Substitution of variables

We assume that $x=\frac{N}{K}, y=\frac{hP}{K}, a=\frac{k}{h}, d=\frac{D}{K}$ .So, we can simplify the system as

$$\begin{cases} x'(t) = rx(1-x)-\frac{axy}{x+d} \\ y'(t) = sy\left(1-\frac{y}{x}\right) \end{cases} \tag{15}$$

and take $p(x)=\frac{ax}{x+d}$

That way, we can get

$$\begin{cases} x'(t) = rx(1-x) - yp(x) \\ y'(t) = y\left(s\left(1-\frac{y}{x}\right)\right) \end{cases} \tag{16}$$

$$x(0) > 0, y(0) > 0, r, s, N, K, \text{h} > 0.$$

where $x(t)$ is the prey population and $y(t)$ is the predator population.So what is a prey population and how does it behave? The prey population is a Logistic growth with the increase of K(K is called capacity), and it has an intrinsic growth rate r in the absence of predators that means the prey grows independently.The predator consu- mes the prey at the rate of functional response $p(x)$,and it's Logistic growth. In this system, its intrinsic growth rate is s,N represents the capacity proportional to the bait, h is the amount of prey consumed by a single predator when $y = \frac{x}{h}$.

② (16) is discretized by Euler Method to obtain:

$$\begin{cases} x_n = x_{n-1} + \delta(x_{n-1}(1-x_{n-1}) - \frac{ax_{n-1}y_{n-1}}{x_{n-1}+d} \\ y_n = y_{n-1} + \delta\left[by_{n-1}\left(1-\frac{y_{n-1}}{x_{n-1}}\right)\right] \end{cases} \tag{17}$$

δ is the step size in (17).

③The mapping of (17) is

$$\begin{cases} x \to x + \delta\left[x(1-x) - \frac{axy}{x+d}\right] \\ y \to y + \delta\left[by(1-\frac{y}{x})\right] \end{cases} \tag{18}$$

If we consider its own growth retarding effect, we can get

$$\begin{cases} x_1(t) = r_1 x_1\left(1 - \frac{x_1}{K_1} - N_1\frac{x_2}{K_2}\right) \\ x_2(t) = r_2 x_2(-1 + N_2\frac{x_1}{K_1} - \frac{x_2}{K_2}) \end{cases} \tag{19}$$

by (18).

Next, we use Matlab to solve the differential equations:we get the rail line of $x(t)$, $y(t)$, and $y(x)$ as fllowing

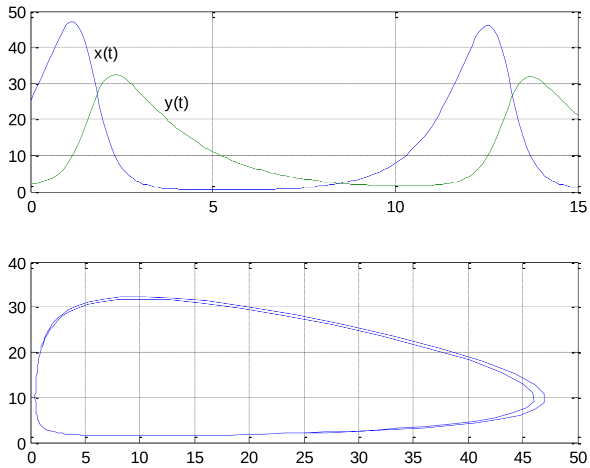

Figure 3: The rail line of $x(t)$, $y(t)$, and $y(x)$

According to Figure 3, it can be predicted that both prey population $x(t)$ and predator $y(t)$ are periodic functions with time t, and we can see that the derailed line $y(x)$ is a closed curve.

4. Summary

This article simply introduces the discrete dynamic system,and describes mainly the Euler Method and its application in discrete dynamic systems.In the application of specific practical problems, the autonomous Lorenz system, and the two-dimensional non-autonomous Holling-Tanner system are considered.We use the Euler Method to discretize them and get the discrete dynamical system form.At the same time, we use Matlab software to do numerical calculation of these dynamic systems and draw the corresponding track graph when the parameters are fixed.

With the further study of the problem, we can do such work: based on the previous work, we can study the equilibrium point or stable point, branch point of the Lorenz system and the two-dimensional non-autonomous Holling-Tanner predator-prey system.Specifically, we can analyze Figure 2 and Figure 3, and then study the bifurcation problem of discrete dynamic system.In general, we can also specifically consider the truncation error of each example, so as to enrich the application of Euler Method in the example.

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