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Analytical solution of advection diffusion equation in two dimensions using different shapes of wind speed and eddy diffusivity

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Abstract: The diffusion equation has been derived in two dimensions using two methods: variable separation and substituting. We compared the results from these solutions to the results from the Copenhagen experiment, taking into consideration the differences in wind speed and eddy diffusivity.

1. Introduction

The industrial revolution has changed living on Earth since then. Pollutants and radioactive isotopes are released into the environment. They play a significant part in environmental degradation, whether they are industrial regions or power plants. Forecasting the behaviour of pollutants from power plants or chimneys, as well as calculating their concentrations, are critical for monitoring air quality, and thus many studies have been dedicated to current analytical and numerical models of pollution simulations in the atmosphere. [1]

Atmospheric dispersion models are mostly based on the gradient transport theory (K), which assumes that the turbulent flow concentration is proportional to the mean concentration gradient, and this theory, along with other assumptions, leads to the horizon propagation equation, which was and is still widely used in analytical and numerical models of pollution dispersion. The method of variable separation was used to create an analytical solution to the two-dimensional atmospheric diffusion equation. In addition, the Fourier transform and square complement methods were used to address the integration problem by considering wind velocity as a function of both downwind distance from the pollution source and vertical height, a semi-analytical solution integral to the atmospheric propagation equation given by [10].

A hypothetical dispersion of pollutants emitted from the urban pollution source in the presence of range winds in the atmosphere showed unstable boundaries. The results show that the mid-range winds generated by the urban heat island blow the pollutants upward, which increases the severity of the urban air pollution [4]. Instead of the variable separation technique, we use the Hankel transform to solve the two-dimensional steady-state fluctuation propagation equation. Following the work of [15]. We determined the parameters of vertical vortex diffusion for the energy law as a function of both the vertical and the wind velocity profile of the energy law to find the integral concentration of normal crosswinds.

The Hankel transform has an advantage over the variable separation technique in that it does not assume a separable solution, as it removes some of the restricts of the form of the solution and introduces a solution in the form of a product of the first-type modified exponential Bessel function that does not include an infinite amount [8]. However, none of these provides a structured approach to finding a solution with generalized functional forms of wind velocity and eddy diffusion.

It is important to mention that the solution to the advection-diffusion equation can be written [11]. These special solutions also satisfy other assisting equations that are easy to solve. These equations are known as differential equations and behave as restrictions on the general solution. If one is concerned about practical applications, in some areas, the most important question about differential limitations is whether it is really necessary to find a general solution to the differential equation before applying restrictions, such as border conditions. However, none of these provides a structured approach to finding a solution with generalised functional forms of wind velocity and eddy diffusion.

The integral concentration of crosswinds was calculated using the Laplace Transform technique and considered wind speed to be dependent on vertical height, whereas vortex propagation is dependent on wind direction and vertical distances [9]. A new semi-analytical approach to solve suspended sediment transport in channels under standardized conditions using the Integrated Circulating Transfer Technology (GITT) by [16].

In this work, the advection-diffusion equation has been solved in two dimensions using two methods, namely, separation of variables and substitution in three models, taking into account the difference in wind speed and eddy diffusion and comparing the results obtained from these solutions with the results obtained from the Copenhagen experiments.

2. Description forms

The mathematical formulation of air pollution dispersion is founded on the conservation of mass equation, which explains advection, turbulent diffusion, and chemical reaction. The advection-diffusion equation is written as follows [13]:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = \frac{\partial}{\partial x}\left(k_x\frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z\frac{\partial c}{\partial z}\right) + R \tag{1}$$

C is the concentration of an air pollutant at any location (x, y, z); R is the removal/reaction term; u, v, and w are the wind components in the downwind (x), crosswind (y), and vertical (z) directions, respectively; and Ky and Kz are the eddy diffusivity coefficients in the x, y, and z directions, respectively.

The physical processes of a material with advection-diffusion are described by Equation (1). Advection transports pollutants in the direction of the wind, whereas diffusion transports pollutants by random movement from a high concentration area to a low concentration region. Turbulence and convection produce these random movements. R denotes removal terms that are ignored, so that x, y, and z indicate the horizontal distance from the source (m), lateral distance from the source (m), and vertical distance above the source (m), respectively.

Thus, turbulent diffusivities K_x, and $K_{v.}$ are increasing functions of averaging time, and $\frac{\partial}{\partial}k_y$ and $\frac{\partial}{\partial z}k_z$represen turbulent material diffusion in the y and z directions. Horizontal advection is much greater than horizontal diffusion, i.e. $u\frac{\partial c}{\partial x} >> \frac{\partial}{\partial x}k_x$, horizontal diffusion is neglected.

The x-axis is oriented in the direction of the mean wind, thus the vertical z is in the direction of w and the y-axis is in the direction of v wind. The velocity components are much smaller than that in the x-direction u, so the two terms $v\frac{\partial c}{\partial y}$ and $w\frac{\partial c}{\partial z}$ can be neglected. Consider a steady-state condition (i.e. $\frac{\partial c}{\partial t} = 0$).

Therefore, Eq. (1) is reduced to the time-independent advection-diffusion equation in three dimensions.

$$u\frac{\partial c(x,y,z)}{\partial x} = \frac{\partial}{\partial y}\left(k_y\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z\frac{\partial c}{\partial z}\right) \tag{2}$$

$$0 < z < h \qquad ,0 \ < y > L_y$$

Where h is the height of the planetary boundary layer, L_y is a large distance in a crosswind direction. The crosswind-integrated concentration $(\bar{C}_y)$ is obtained by integrating equation (2) with respect to y from $-\infty$ to $+\infty$ as follows:

$$u\frac{\partial C_y(x,z)}{\partial x} = \frac{\partial}{\partial z}\left(K_z\frac{\partial C_y(x,z)}{\partial z}\right) \tag{3}$$

The Crosswind Integrating Concentration at height z is represented by$C_y(x, z)$.
Equation (3) is subjected to the following boundary conditions:
Mass continuity is achieved at x = 0.

$$u(z)\, C_y(x, z) = Q\,\delta(z-h) \qquad \text{at } x=0 \tag{4}$$

The crosswind integrating concentration vanishes at x, and z tends to zero.

$$C_y(x,z) = 0 \text{ at } x,\ z \rightarrow 0 \tag{5}$$

The flux in the vertical direction equals zero at the ground surface ("z=0") and the height of the planetary boundary layer (PBL) of the air pollutant ("h").

$$\partial\, C_y(x,z)\,/\partial z = 0 \quad \text{at } z=0,\ h \tag{6}$$

The crosswind integrating concentration is achieved at x, and z tends from ∞ to zero.

$$\int_0^z \int_0^x \bar{C}_y(x,z)\ \partial z\,\partial x = Q_L \quad \text{at} \quad z=h,\ x=L_x \tag{7}$$

5-The flux in the horizontal direction equals zero at x=0 and at a large distance Lx in the x-direction.

$$\partial\,\bar{C}_y(x,z)\,/\partial z = Q_L \quad \text{at} \quad x= L_x \tag{8}$$

Where h, Q, Lx, u (z) and δ represents the height of the planetary boundary layer (PBL), the source strength or emission rate (g/s^3), the large distance in the horizontal direction (7000 m), downwind speed (m/s), and Dirac delta function, respectively. Q_L (mass per unit length divided by unit time) (10 mgm^2 sec^1) is the continuous-point source strength [7].

Equation (3) can be solved in three models as follows:

2.1 The initial model

Wind speed and vertical eddy diffusivity are taken as follows [7]:

$$u(z) = \frac{u_*}{0.4}\left(ln\left(\frac{z}{z_0}\right) + \frac{5z}{L}\right), \quad K_z = 0.4u_*\, z \tag{9}$$

Where, roughness length (z_o), friction velocity ($u_*=0.1u\ (z)$), vertical eddy diffusivity (K_z) and vertical height (z), Monin-Obukhov length (L) [12].

Substituting Eq. (9) into Eq. (3), we obtain [H1]:

$$\frac{u_*}{0.4}\left(ln\left(\frac{z}{z_0}\right) + \frac{5z}{L}\right)\frac{\partial C_y(x,z)}{\partial x} = 0.4u_*\, z\frac{\partial^2 C_y(x,z)}{\partial z^2} + 0.4u_*\frac{\partial C_y(x,z)}{\partial z} \tag{10}$$

The general solution to Eq. (10) is as follows:

$$\frac{C_y(x,z)}{Q} = \frac{Q_L(Q_L-1)\left(ln\left(\frac{h}{z_0}\right)+\frac{5h}{L}\right)}{L_x\left[Exp - \frac{(Q_L-1)}{(0.4)^2 L_x}\left(ln(z_0)ln(h) - \frac{5h}{L} - \frac{(ln(h))^2}{2}\right)\right]}\ \frac{\left[\frac{1}{2\pi}+\frac{1}{\pi}(cos(z-h))\right]}{\left(ln\left(\frac{z}{z_0}\right)+\frac{5z}{L}\right)}\left[Exp\left(-\frac{(1-Q_L)}{L_x}\right)x\right]\left[Exp - \frac{(Q_L-1)}{(0.4)^2 L_x}\left(ln(z_0)ln(z) - \frac{5z}{L} - \frac{(ln(z))^2}{2}\right)\right] \tag{11}$$

At n = 0, a Fourier series expansion gives [2].

2.2. The second model

Taking the wind velocity and the vertical eddy diffusivity from [14]:

$$u(z) = u_1 z^m,\ K(z) = K_1 z^n\ ,\ \frac{\partial K_z}{\partial z} = nK_1 z^{n-1}\ \text{ and } \frac{\partial K_z}{K_z \partial z} = \frac{nK_1 z^{n-1}}{K_1 z^n} = \frac{n}{z} \tag{12}$$

Where u1 (10 m/s) and K1 (10 m2/s) but m and n are the values of the constants.

Substituting from Eq. (12) into Eq. (3) as follows:

$$u_1 z^m \frac{\partial C_y(x,z)}{\partial z}\frac{\partial z}{\partial x} = K_1 z^n \frac{\partial}{\partial z}\left(\frac{\partial C_y(x,z)}{\partial z}\right) + nK_1 z^{n-1}\left(\frac{\partial C_y(x,z)}{\partial z}\right) \tag{13}$$

The general solution of Eq. (13) is as follows:

$$\frac{C_y(x,z)}{Q} = \frac{\left[\frac{1}{2\pi}+\frac{1}{\pi}(cos(z-h))\right]\ x\ k_1(m-n+2)}{(u_1)^2\ z^{2m-n+1}}\left\{\left[Exp\left(-\left(\frac{u_1 z^{m-n+2}}{xk_1(m-n+1)(m-n+2)} + \frac{n^2\ ln(z)}{z}\right)\right)\right]\right\} \tag{14}$$

2.3. The third model

The downwind speed and vertical eddy diffusivity are taken from [3] as follows:

$$u(z) = \frac{u_*}{0.4}ln\left(\frac{z}{z_0}\right),\ K_z = \frac{0.4u_*\, z}{0.47\left(1-\frac{9z}{L}\right)^{-1/2}}\ \ and\ \ \frac{\partial K_z}{\partial z} = \frac{A(2L-27z)}{2\sqrt{L}\sqrt{L-9z}} \tag{15}$$

Substituting from Eq. (15) in Eq. (3) as follows:

$$\frac{u_*}{0.4}ln\left(\frac{z}{z_0}\right)\frac{\partial C_y(x,z)}{\partial x} = \frac{0.4u_*\, z}{0.47\left(1-\frac{9z}{L}\right)^{-1/2}}\frac{\partial}{\partial z}\left(\frac{\partial C_y(x,z)}{\partial z}\right) + \frac{0.4u_*(2L-27z)}{2(0.47)\sqrt{L}\sqrt{L-9z}}\left(\frac{\partial C_y(x,z)}{\partial z}\right) \tag{16}$$

The general solution to Eq. (16) is as follows:

$$
\frac{C_y(x,z)}{Q} = \frac{Ln\,(L_x)\,Q_L\left(\frac{2\ln z_0}{1+\left(\sqrt{\frac{(9h-L)}{L}}\right)^2} - \frac{L\,(9h-L)-9(\ln(h))^2}{(9h-L)} - \frac{(9h-L)}{L} + \frac{3}{(9h-L)}\right)}{x\left(\frac{2\ln z_0}{1+\left(\sqrt{\frac{(9z-L)}{L}}\right)^2} - \frac{L\,(9z-L)-9(\ln(z))^2}{(9z-L)} - \frac{(9z-L)}{L} + \frac{3}{(9z-L)}\right)} \tag{17}
$$

$$
\left(\left\{\begin{array}{l} Exp - \left\{\frac{0.47}{(0.4)^2}\left[\left(2ln(z_0)\ tan^{-1}\sqrt{\frac{(9z-L)}{L}}\right) - \frac{2(ln(z))^2}{\sqrt{(9z-L)}}\right] - ln\left(\frac{z}{(9z-L)}\right) + 3\,ln(9z-L)\right\} + \\ \left\{\frac{0.47}{(0.4)^2}\left[\left(2ln(z_0)\ tan^{-1}\sqrt{\frac{(9h-L)}{L}}\right) - \frac{2(ln(h))^2}{\sqrt{(9h-L)}}\right] - ln\left(\frac{h}{(9h-L)}\right) + 3\,ln(9h-L)\right\} \end{array}\right\}\right)
$$

3. Experiment result

To test the results of the three models to find the concentration of pollutants on the earth's surface and compare these data with the data from the Copenhagen experiments [5.6], the pollutants were released at a height of 10 metres above the ground and the vertical height was 0.06 meters, which is close to the ground's surface, using the three experimental models as described by Eqns. (17). This can be evaluated by examining the findings in Table 1 and Fig.1, which show that Eqns. (11), (14), and (17) are all within a factor of two.

4. Calculation results and discussion

Table 1: Meteorological parameters and concentrations measured during the Copenhagen Experiment

Run.	H (m)	X (m)	m	n	-L(m)	u (m/s)	$C_y(x,z)/Q$ (10^{-4} sm^{-3})			
							Observed	Eq.(11)	Eq.(14)	Eq.(17)
1	1980	1900	0.90	0.10	37	3.03	6.4800	6.4920	6.3350	6.056
1	1980	3700	0.80	0.20	37	3.03	2.3100	2.3100	2.3480	1.99
2	1920	2100	0.70	0.30	292	3.03	5.3800	5.3600	4.7300	5.66
2	1920	4200	0.60	0.40	292	7.99	2.9500	2.7560	2.8900	3.75
3	1120	1900	0.50	0.50	71	7.99	8.2000	7.8210	8.3670	7.45
3	1120	3700	0.40	0.60	71	3.46	6.2200	6.0235	5.0640	6.039
3	1120	5400	0.30	0.70	71	3.46	4.3000	3.5280	3.9500	2.59
4	390	4000	0.20	0.98	133	3.46	6.7200	5.7940	6.5530	5.804
5	820	2100	0.10	0.90	444	4.08	4.1240	4.7000	5.5850	5.059
5	820	4200	0.10	0.10	444	5.05	4.9700	5.2540	4.2640	5.087
5	820	6100	0.90	0.10	444	5.05	3.9600	4.1750	2.7980	2.056
6	1300	2000	0.80	0.20	432	5.05	2.2200	2.8960	1.9430	2.431
6	1300	4200	0.70	0.30	432	11.73	1.8300	1.6420	2.4780	1.016
6	1300	5900	0.60	0.40	432	11.73	6.7000	6.3710	6.1190	5.065
7	1850	2000	0.50	0.50	104	11.73	3.2500	2.7320	3.7780	2.257
7	1850	4100	0.40	0.60	104	5.91	2.2300	2.3640	2.4800	2.072
7	1850	5300	0.30	0.70	104	5.91	4.1600	4.3280	4.0990	3.741
8	810	1900	0.20	0.98	56	5.91	2.0200	2.0740	1.8950	1.262
8	810	3600	0.20	0.98	56	7.73	1.5200	1.5790	1.1620	1.581
8	810	5300	0.10	0.90	56	7.73	4.5800	4.5140	4.2570	1.435
9	2090	2100	0.10	0.90	289	7.73	3.1100	2.5440	3.0390	3.53
9	2090	4200	0.20	0.89	289	8.31	2.5900	2.0660	2.0460	2.035
9	2090	6000	0.10	0.90	289	8.31	6.4800	5.2750	5.8620	6.256

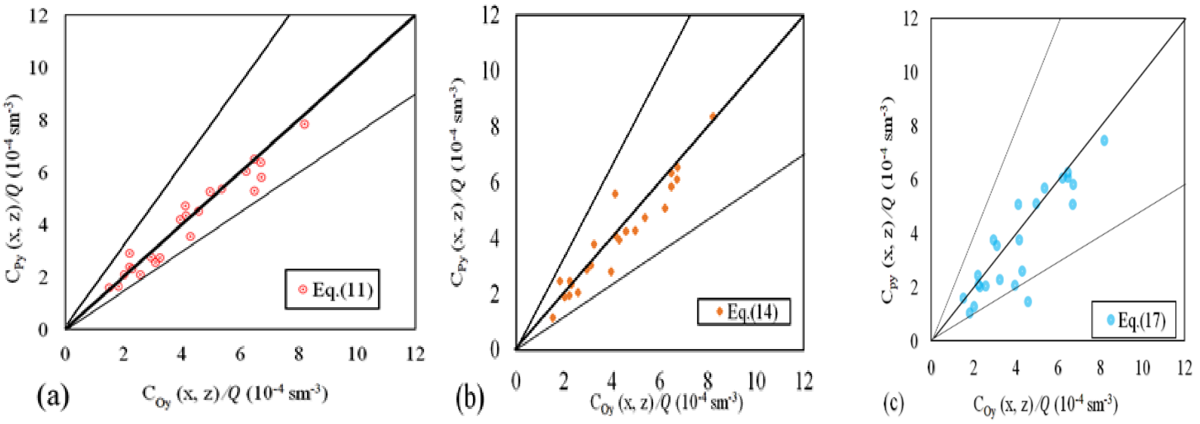

Figure 1: Observed (CO) and predicted (CP) crosswind integrating concentration with emission rate during Copenhagen experiment for Eqns. (11), (14), and (17).

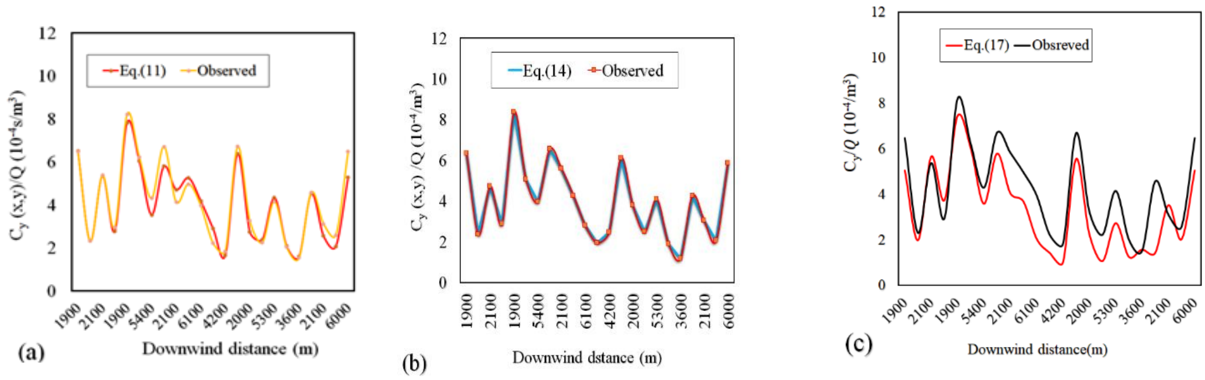

Figure 2: Comparisons between Downwind Distance and concentrations during Copenhagen experiment for Eqns. (11), (14), and (17).

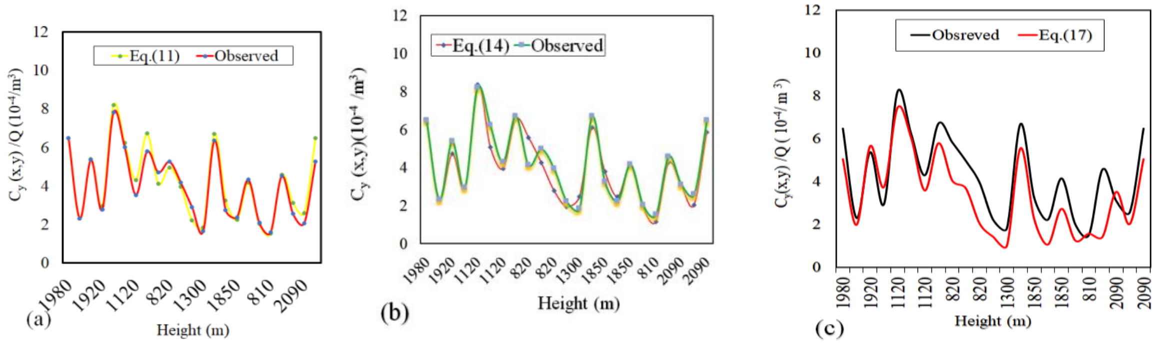

Figure 3: Comparisons between height and concentrations during Copenhagen Experiment Eqns. (11), (14), and (17).

Figures 1 show that Eqns. (11), (14), and (17) are within a factor of two.

Figures 2 and Figure 3 show that the predicted concentrations are better than the observed concentrations.

5. Conclusion

The advection-diffusion equation has been solved in two dimensions using two methods, namely, separation of variables and substitution in three models, taking into account the difference in wind speed and eddy diffusion, and compared the results obtained from these solutions with the results obtained from the Copenhagen experiments, showing that (11), (14), and (17) are within a factor of two of the observed data.

Authors' contributions

The experiments were conducted by Khaled S. M. Essa; the experiments were conducted and analyzed by Sawsan S. M. El Saied, and the manuscript was reviewed by all authors.

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