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HPM-based analysis of lessons with heterogeneous of the first n terms and the sum of the equal ratio sequence

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Abstract: 'The first n terms and the sum of the equal ratio sequence' is a second-year high school textbook in the Shanghai edition, and many teachers have already designed their teaching of this content. This article uses the HPM lesson review framework to compare and analyse two teachers' designs for teaching 'The first n terms and the sum of the equal ratio sequence', in order to find out the merits of integrating history into mathematics classroom teaching, and to provide teachers with references for designing future mathematics teaching processes based on the history of mathematics.

1. Introduction

With half of HPM knowledge coming from the history of mathematics (H) and the other half from classroom teaching (P), HPM research has moved from a theoretical 'top-down' approach to an HPM perspective on educational design and practice. [1] As Jankvist suggests, the use of history of mathematics in teaching and learning can be approached in two ways: using history as an affective tool to motivate students; and using history as a cognitive tool to support the learning of mathematics. [2].

In the Shanghai version of the textbook, 'The first n terms and the sum of the equal ratio sequence' is located in Mathematics 5, Chapter 2, Section 5. Based on similar backgrounds in the history of mathematics, two teachers from different secondary schools in Shanghai, Teacher A and Teacher B, tailor their teaching designs according to their own understanding and students' learning situations. In this paper, the similarities and differences between the two lessons in terms of the selection of history of mathematics materials, the way they were incorporated and the effectiveness of their use are compared and analysed using the HPM analysis framework of the lesson with heterogeneous, with a view to providing a reference for HPM lesson studies.

2. Historical material

The problem 79 recorded in the Rhind papyrus originating in ancient Egypt shows that the ancient Egyptians had already concluded the relationship between the sum of the first n terms of an isoperimetric series and the sum of the first $n-1$ terms. By the 9th century, the Indian mathematician Mahavira gave the formula for the summation of an isoperimetric series in his book, which was derived along the same lines as the ancient Egyptian priests. But it is easy to see that the ancient Egyptians used in their derivation the familiar recursive ideas of today.

In the 18th century, in his Foundations of Algebra, Euler used the dislocation subtraction method for the derivation of summation formulas. He discussed the general formula for summing equipartite series in the order of the special series $1, 2, 4, \cdots, 2^n, \cdots$ to the series $1, 3, 9, \cdots, 3^n, \cdots$ and finally to $a, aq, \cdots, aq^{n-1}, \cdots$, derived the general formula for finding the sum of the geometric progression.

Euclid gives a completely new derivation of the formula for the sum of the geometric progression in Proposition 35, Volume 9 of Geometry Originally. First, the definition of an isoperimetric series gives

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{a_{n+1}}{a_n}, \tag{1}$$

And by the fractional ratio

$$\frac{a_2 - a_1}{a_1} = \frac{a_3 - a_2}{a_2} = \cdots = \frac{a_{n+1} - a_n}{a_n}. \tag{2}$$

The combined ratio then gives

$$\frac{a_{n+1} - a_1}{a_1 + a_2 + \cdots + a_n} = \frac{a_2 - a_1}{a_1}, \tag{3}$$

i.e.

$$\frac{a_1 q^n - a_1}{S_n} = q - 1. \tag{4}$$

Thus, when $q \neq 1$, we have

$$S_n = \frac{a_1\left(1 - q^n\right)}{1 - q}. \tag{5}$$

The ancient Egyptians, on the other hand, used recursion to derive the formula for finding the sum of the geometric progression. Since

$$a_1 + a_2 + \cdots + a_n + a_{n+1} = a_1 + q\left(a_1 + a_2 + \cdots + a_n\right), \tag{6}$$

i.e.

$$S_n + a_{n+1} = a_1 + qS_n, \tag{7}$$

This gives the formula for the first n terms and the sum of the equal ratio sequence in the general case, i.e. when $q \neq 1$.

In his Fundamentals of Algebra, the French mathematician Lacroix discovered an alternative method - Head to tail method, from

$$S_n = a_1 + a_1 q + \cdots + a_1 q^{n-2} + a_1 q^{n-1} \tag{8}$$

We get two variant forms

$$S_n - a_1 = a_1 q + \cdots + a_1 q^{n-2} + a_1 q^{n-1}, \tag{9}$$

$$S_n - a_1 q^{n-1} = a_1 + a_1 q + \cdots + a_1 q^{n-2} \, , \tag{10}$$

And by comparing the two equations we get

$$S_n - a_1 = q\left(S_n - a_1 q^{n-1}\right) \; . \tag{11}$$

These are the materials from the history of mathematics selected and used by the two teachers in the HPM lesson study.

3. Macro comparison of the two sessions

3.1 Teaching objectives and key points

The objectives set by Teacher A and Teacher B are similar: (1) to understand and master several methods of deriving the sum of the geometric progression, and be able to use the formula to solve some simple problems; (2) to appreciate the mathematical ideas and methods used in the process of deriving the formula; (3) to train students to look at the problems in life with a mathematical perspective, and to stimulate their spirit of exploration. The difference between the two is that: Teacher A poses a question to students in the form of a video, and then goes straight to the topic to explore the derivation of the formula after reviewing the relevant knowledge, and finally, the story of strangers and rich people is used to infiltrate moral education into students; Teacher B shows students the general ideas and methods of derivation of the formula after giving the problem situation, and leads students to conduct group investigations, and finally summarises students' methods.

Both teachers also have essentially the same teaching focus and difficulties.

Teaching focus: the derivation of the formula for the first n terms and the sum of the equal ratio sequence and its simple applications.

Teaching difficulties: the understanding of the method of deriving the formula for the first n terms and the sum of the equal ratio sequence.

3.2 Teaching process

The teaching process of both lessons can be divided into four parts: introduction to the situation, investigation and conclusion of the formula, application of the formula and summary of the lesson. See Table 1 for more details.

As can be seen from Table 1, both Teacher A and Teacher B base their teaching design on the HPM perspective. The similarities lie in the fact that they both demonstrate a variety of ancient and modern mathematical methods, integrating the history of mathematics into their teaching and allowing learners to experience the process of mathematicians' exploration. The differences are that Teacher A introduces the problem situation more naturally by presenting the story of the rich man from everyday life; in the concluding reflection stage, students' views on the main character of the problem are collected, which helps students to communicate with each other and improve their moral character. Teacher B provides a variety of ways to prove the formula for finding the sum of the geometric progression in the history of mathematics before students proceed with their investigations, which helps problem solvers construct their understanding of the problem and provides learners with thinking tools.

Table 1. Teaching sessions for Teacher A and Teacher B

Sessions	Teacher A	Teacher B
Introduction to the situation	The lesson begins with students watching a video - 'The story of the rich man and the stranger'. The teacher takes students through the definition of the isoperimetric series, etc. and asks how the formula for the sum of the first n terms of an isoperimetric series looks like. By analogy with the arithmetic series, suppose the first term a_1 and the common ratio q of an isoperimetric series $\{a_n\}$ are known and how its sum of the first n terms is expressed.	The teacher guides students to recall the content of the arithmetic series by first throwing out the wife problem from Scholars' Arithmetic, then introducing the story of the king's reward for wheat. Students review the definition of the isoperimetric series, etc. The teacher asks the question: how to represent S_n in terms of the basic quantities a_1, q, n.
Investigation and conclusion of the formula	The teacher guides students through the equation method, head to tail method, proportional method and dislocation subtraction method, and gives a brief history of them.	The teacher creates a micro-video before class that introduces the general idea of the derivation of the formula and then pushes it out to students via the internet. Students are guided to conduct group investigations in class. After the group discussion, the teacher sends a representative from each group to the board to present each group's method of derivation.
Application of the formula	Introduce the sum of finite terms of isometric sequence with questions from “The Mathematics in Nine Sections”, using Examples 1 and 2 for students to consolidate.	After deriving the formula, students are asked to apply it to solve the problem of the King's reward for wheat. The teacher then asks students to solve ancient and modern problems with the formula to further appreciate the need and usefulness of learning this formula.
Summary of the lesson	Students summarize their takeaways on their own, and the teacher suggests that The Story of the Rich Man and the Stranger is left for students to think about.	Students summarise their gains independently and the teacher leaves question 4 of the Chinese Han Dynasty's “The Mathematics in Nine Sections” for students to think about after class.

4. Micro comparison of the two sessions

4.1 Suitability of historical material

Five principles should be considered when selecting historical materials for teaching mathematics: Interesting, learnability, effective, humanity and scientific. [3]

The historical materials used by the two teachers in the teaching process, such as the ‘ method of transplacement subtraction’ and the ‘ Head to tail method’, have historical origins and are in line with the principle of scientific. The use of these historical materials helps to achieve ‘understanding and mastery of several methods of deriving the first n terms and the sum of the equal ratio sequence’, ‘appreciation of the mathematical ideas and methods used in the derivation of formulas’, and ‘cultivate students to look at the problems in life from the perspective of mathematics’. The objectives are in line with the principle of effective. Both teachers took into account the cognitive base of students in the teaching process, designed the inquiry process from the students' perspective and introduced the derivation methods of a number of mathematicians so that students could appreciate the unique charm of mathematics in the course of historical development, which is in line with the principles of learnability and humanism. The questionnaire collected after the lesson showed that the majority of students found Teacher A's lesson a good or even very good experience, and 80% of the students said they liked the way the history of mathematics was integrated into the teaching of mathematics, which is in line with the principle of Interesting.

4.2 The naturalness of integration

Teacher A showed students the story of the rich man and the stranger, from the definition of geometric sequence, the common term formula of geometric sequence to the summation formula of geometric sequence， giving students the opportunity to explore on their own, arousing their interest in the derivation of the formula on the basis of their original knowledge structure, in line with the psychological sequence of learners. On the other hand, the order in which the methods are introduced: equation method → head to tail method → proportional method → dislocation subtraction method, although in line with the students' cognitive order, is not quite in line with the historical order.

Teacher B itself is an inquiry lesson, using the ‘context - problem - inquiry’ teaching model, starting from the wife's problem and the King's Reward wheat problem, which the students can visualise. The students are guided to think on their own and to be courageous in their investigations. Starting from the sum of finite terms of geometric sequence， the students move from ' hard calculation' to the derivation of a formula in a logical sequence. The teacher summarises the results of the students' investigations by comparing them with the mathematical methods in history, although some of the methods obtained by the students are consistent with those of mathematicians in the past, but this is not very consistent with the historical investigation of the formula for the summation of isoperimetric series, which does not conform to the historical sequence.

In summary, the integration of the history of mathematics was more natural and more explicit in both Teacher A's and Teacher B's classrooms.

4.3 Pluralism of approach

The main way in which Teacher A uses the history of mathematics in her teaching is in the form of replication. His ‘Nine Chapters of Arithmetic’ in the application of formulae was a replication, showing students the wisdom of the ancients, while the ‘Story of the Rich Man and the Stranger’ did not involve the history of mathematics. Teacher B's use of the history of mathematics is mainly replicative and responsive, with the wife problem and the king's wheat problem used in the introduction of the situation being appropriately simplified to increase students' opportunities for inquiry, while the 'Nine Chapters of Arithmetic' in the post-lecture reflection is taken directly from history, the former being responsive and the latter being replicative. Neither of them used a reconstructive approach, for example, but in contrast, Teacher B used a richer selection of mathematical history than Teacher A.

4.4 The profundity of values

The four derivation methods included in Teacher A's teaching use the idea of equations, which takes into account the students' cognitive development level, allowing them to construct new knowledge based on their existing knowledge and allowing them to accept new mathematical methods more naturally, which reflects the 'harmony of knowledge'. The variety of derivation methods and the impressive mathematical drama infected the students, who were deeply immersed in the process of investigation under the guidance of the teacher, and the process of investigation by mathematicians made the students more determined to pursue truth and innovation, reflecting the 'effect of moral education'. The teacher not only introduces the derivation process of methods such as head to tail method, dislocation subtraction method, but also describes the information about the mathematicians behind them, giving students a sense of the origin and flow of mathematical knowledge. There are also comparisons between ancient and modern methods, and students are able to appreciate the historical similarities in the learning process of mathematics, which reflects the 'charm of culture'.

In Teacher B's teaching, the task is not only to master the equal ratio sequence sum formula, but also to appreciate the mathematical methods involved in the investigation of the formula. Students learn a variety of mathematical methods in the classroom, and these gems of history help to broaden students' thinking and make them feel the beauty of mathematics, reflecting the 'beauty of method' of this teaching design. Group work helps students to understand mathematical topics, and their attitudes and beliefs about mathematics become more positive as they explore methods of derivation. In this way, the 'joy of inquiry' is successfully achieved as students experience the ultimate success of formulae derivation. In the process of formula investigation, common mathematical ideas such as from special to general, equation thought, number shape combination thought and other common mathematical thoughts are embedded, in addition to the four mathematics core literacy of mathematical modelling, logical reasoning, mathematical operations and intuitive imagination, which reflect the ‘ improvement of ability’.

5. Conclusions

From the above analysis, we can understand that, from the perspective of the appropriateness of historical materials, the selection of historical materials in both lessons basically conforms to the principles of interesting, learnability, effective, humanity and scientific. In terms of the naturalness of integration, both teachers' teaching designs conformed to the psychological and logical order, but did not quite fit the historical order. In terms of the plurality of methods, Teacher A used the replicative style, while Teacher B used the replicative and responsive styles, neither of them using the reconstructive style. In terms of the profundity of values, both lessons reflect the 'harmony of knowledge', the 'beauty of method' and the 'improvement of ability ', with Teacher A's teaching also reflecting ‘Teacher A's teaching also reflects ' effect of moral education ' and 'the charm of culture'.

The following insights can be gained from the comparison and analysis of the two lessons.

Firstly, appropriate historical materials should be selected. In the selection of historical materials, both the students' existing knowledge base and the sequence in the advancement of history need to be taken into account. A teaching design that conforms to the students' psychological sequence, as well as the logical and historical sequence, can bring out the greatest charm of the history of mathematics.

Secondly, the time spent on inquiry should be set at a reasonable level. For example, Teacher B's teaching, although it stimulates students' creativity and enriches their inquiry experience, is difficult to implement successfully in daily teaching because its inquiry time is too long. And when presenting the results of group work, appropriate time should be set aside to accommodate other students to express their views in order to encourage mutual learning and progress among students. In addition, when summarising, teachers should allow some time to sort out the connections between the various methods so that they can be transferred to students in their future learning and life.

After that, teachers should control the ‘points of occurrence’ in the teaching process in order to achieve effective teaching. When introducing a situation, teachers should create a reasonable 'point of conflict'. When guiding students to experiment independently, teachers should design ‘points of inspiration’. When summarising, teachers should facilitate the production of 'internalisation points'. When setting aside post-lesson reflection, the teacher should explore further 'deepening points'. [4-5]

Finally, the focus should be on the infiltration of moral education in the curriculum design. Teacher A portrays two very different characters in her teaching: the wealthy man who is greedy for money and the stranger who has good mathematical skills. During the teaching process, students are able to fully appreciate the value of deep mathematical literacy in their lives, which helps to strengthen their mathematical beliefs. In addition the process of inquiry, which involves constant mapping, moving forward and reflecting, helps students to develop critical thinking and problem-solving skills.

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