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A simple calculation model for transport of visitors on a closed circuit

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Abstract: It is examined a very simple, geometrically closed configuration concerning transport systems. The problem analyzed is a multi-objective optimization problem, in which an ever increasing set of visitors is urged to visit and move from a given set of sites on a closed path. An essentially kinematic approach is developed and the performance evaluation is obtained by means of a combinations of variables. It is also examined a particular realization in which one makes use of $\boldsymbol{C_0}$-functions instead of discrete variables, in the spirit of classical mathematical physics. Some features of the model show relevant differences with others concerned with traffic and transport problems, for the presence of Diophantine integral evolution equations in place of statistical and/or numerical complex evaluation methods. This work is the first part of a more thorough discussion on dynamical equations in transport systems, including simulations and optimization schemes.

1. Introduction

As universally acknowledged, big and small cities, metropolitan areas and little urban agglomerations all are besieged by growing patterns of traffic congestion[1]. In very succinct words the problem is strictly related and overwhelmingly caused[2] by excessive car dependency, resulting in fact in reduced mobility and sustainability almost everywhere[3]. Since the birth of such a problem, city administrations ever wasted lot of time and money to solve this and related environment problems. The aim is looking for implementing alternative intermodal transportation systems[4]. The purpose would be to minimize all sort of negative, congestion effects due to private car usage while enhancing possibly economic viability and the quality of life[5]. In such a general scenario cities governments recognized urban public transportation system as a fundamental element in achieving a well behaved transportation alternative system[6]. The provision of an attractive, viable public transit system as a good alternative to private cars is becoming a real fact[7].

In the perspective of urban transit systems it is common practice with buses to regulate the departures from a limited number of stops, by holding the operators on their stop until the scheduled time. More recently, big sets of data and simulations showed[8] nevertheless this control strategy is generally unsuccessful in improving the regularity of the service along the line. Such simulations showed also drivers adjust their speed based on the actual performance objectives. Good indications, on the contrary, come from a control strategy that regulates departures from all stops looking at the headways from the preceding bus and the following bus as well[9].

Fundamental steps to have good insight in transit transportation systems include[10] the Operational Planning Decomposition Process (OPDP). This is a very general planning process adopted in many production systems to generate well integrated and executable procedures able to increase production. It consists[11] of four basic activities usually performed in sequence. These are: (i) the design of the transmission route, (ii) the prescription of the timetable, (iii) the scheduling of the conveyors and (iv) the crew preparation. Important progress and innovation in the field of OPDP was the introduction of automated counting electronic devices and control systems. These are useful to quantify the additional time and costs due to traffic congestion and make sudden improvements during the service itself. The chance of adopting automatic passenger counter (APC) systems in the Operational Process turns out to be a very promising solution. In recent years they have been implemented in public transit systems in order to check bus occupancy, location, travel time and so on[9]. Such kind of information is important for a lot of applications which include performance evaluation, service planning, safety improvements and so on.

However it is not so easy to implement APC systems on any sort of transport system. In some conditions or experimental situations it is hard to apply online control systems or just continuously online. For instance it may be possible to send a control signal only at certain fixed times. In addition, in other situations it is useful to recourse to the large variety of physical effects caused by the passage of a vehicle on a route. This technique gives rise to a correspondingly great variety of sensors and recording techniques[12] which are currently adopted in any experimental set up.

In this paper the transport systems subject has been examined considering a very simple configuration. This configuration, however, seems to be easily applicable and extendible to many, more complex and realistic situations, so that it looks as a scheleton configuration. In this model the adoption of APC is quite limited in time and it takes a lower importance.

The structure of the problem presented here is analyzed from a theoretical point of view, i.e. without the recourse to known experimental data. This choice is essentially given by the mathematical relevance, if any, of the problem shown, before than physical applications. A particularly simple case has been anycase proposed and examined thoroughly. Concerning the APC control, we introduced a simplified version of Control System (CS), acting on the transport system only "as the whole", that is "as it were a rigid structure" in which the relative distances between conveyors remain unchanged. The effect of this Control System is indeed to modify the speed of all the conveyors at regular time intervals, leaving the distances between them on average unaltered.

After the present introduction, in section 2 we present the specific situation we want to analyze, together with some questions associated to the time evolution and it is proposed a kind of solutions. Then in section 3 we discuss the details of the conceptual scheme of the proposed solution and the entire realization. Finally, in section 4 it is exhibited a particular solution obtained in the special configuration given by treating the fundamental quantities under consideration like sets of equally spaced solitons. In the concluding section 5 the main suggestions coming from this work and possible extensions are exposed.

2. Model description

It is generally known that traffic transport systems are very complex dynamical structures, whose construction and evolution are driven by lot of variables and critical parameters of very different species and nature. In quite general terms, this evolution can be grossly described in analytical terms by means of a generally unknown function[13]:

$$y = f\ (\$, T, E, M, A) \tag{1}$$

in which the symbol \$generically indicates costs to be incurred, T represents paths to be covered, E concerns energy altogether consumed, M refers to the amount of mobility required and A takes in consideration the amount of accessibility. Here$y$ stems for the best of performance of the system in terms of an appropriately defined measurable quantity whereas f is a suggestion for a viable search for the best of y. Here we will concentrate exclusively on the components$T$ and M of the previous "generalized function" y , assuming by hypothesis a factorization of f between different contributions; then neglecting questions related to $\$, E$ and A.

Then we start considering a simply connected path C_ζ in a closed region of $\boldsymbol{R}^2$, what we call a circuit in $\boldsymbol{R}^2$. The path C_ζ is designed to connect four strategic points $\{A, B, C, D\} \in \boldsymbol{R}^2$ .The connection is operated by a Transport System consisting in a set of "conveyors". The motion of the conveyors is only restricted to the circuitC_ζ. The strategic point A plays a special role because it represents the unique starting and arriving point in the circuit.

Together with the conveyors we suppose the existence of a set of "conveyed objects". We can see them like visitors of the strategic points. Visitors also are allowed to move only through C_ζ.

The preferred mean of transport of these objects is by conveyors. Nevertheless, if it happens conveyors are missing for a given time, visitors are able to move by themselves. In this case we interpret such a behavior as very similar to drift diffusion from point-like sources. The interpretation of this motion spontaneously issued in terms of diffusion arises from the alleged ignorance of the "objects" concerning the right direction they must take in order to arrive to the next strategic point. Such an ignorance contributes, indeed, to give the incoming motion some chaotic behavior, especially on the starting direction. This looks formally analogous to classical random walk model. Most in particular we believe the sort of "procession" so arising is better represented by means of a drift - diffusion model than a particle - or thermal diffusion model. The setting presented here shows indeed a structure partly probabilistic but also partly deterministic.

The situation is then not a good example of a completely stochastic dynamic systems. The deterministic aspect of the motion arises because each visitor feels in itself from the beginning an intrinsic, nearly constant and uniform need (i.e. "a force") to move at regular times in a well defined direction. This hints to describe it as a drift-diffusion instead of thermal- or concentration- or pressure- diffusion.

Evidently the specific application of our circuit C_ζ with associated "transported objects" can vary greatly, ranging from guided visits to museums or exhibitions to vaporetto service in lagoon areas and even in tourist information services on the internet. Consequently and depending on this choice, the nature of the corresponding Transport System elements will also be different.

In consideration of possible applications to urban cities, it seems useful hereafter to rename the conveyors and the transported objects calling them *"the buses"*, $\{b_j\},\ j = 1,..,N_B$ and *" the pedestrians"*, $\{p_i\},\ i = 1,..,N_P$. Sometimes the pedestrians will be called *"the visitors"*, with the same meaning. As we said, pedestrians hope to move aroundC_ζ by using buses; but due to time reasons, if needed, pedestrians are able to move on C_ζ also without buses, i.e. "by feet". It is mandatory for buses and pedestrians to follow the order $A - B - C - D$ while they are moving around C_ζ. It is also forbidden to the buses and pedestrians to reach any point not consecutive another one unless reaching first all the intermediate ones. Exceptions to this rule will be described in the following.

Buses move with constant speed V for each lap around C_ζ. At the beginning of the next lap the speed can be modified by the Control System, depending on the performance of the transport system itself. The graph of $V\ (t)$ would then be similar to a step function. We also suppose the existence of a diffeomorphic map between C_0-regular representations of C_ζ in arbitrary coordinates and the ordinary circumference$\boldsymbol{C}' \in \boldsymbol{R}^2$.

This assumption allows us to simplify the description of the "true" motion overC_ζ with a circular one.Indeed, if $V\ (t)$is sufficiently constant in time we may describe the motion in terms of an approximately uniform circular motion over$\boldsymbol{C'}$, done with constant angular speed Ω. The only exception to a perfectly uniform circular motion would come by the presence of the four stops required at points $\{A, B, C, D\}$. In other words we translate the motion on C_ζ to an approximately uniform circular one on the unitary circumference having four strategic points, $\{A, B, C, D\}$, located at four special angular coordinates $\theta_i,\ i = A, B, C, D$.

The meaning of "visiting a point" for a pedestrian p_j requires an explanation. We say that p_j visits a point P_i if p_j gets off the bus b_k he were occupying when b_k stopped at P_i. This is different from "passing to point P_j", which doesn't imply *"getting off"* the bus, but only arriving at P_i . Once reached a point, the time required in order to complete the visit is $t_V = const$.

The pedestrians purpose is to start from point A and return there after visiting in sequence the greatest possible number of points B, C, D in the lowest possible time. The goal for the Transport System consists in collecting the greatest possible number of pedestrians to every round trip with a given set of buses. The initial time of the process ist_0. The number of pedestrians conveyed by buses at time t and position θ on $C_\zeta$is $Q(\theta, t)$ so that the maximum bus capacity is $Q_M \equiv Q(\theta, t_0)$ assuming the buses are completely empty at the beginning.

As quoted above, point A plays a special role, since it is the source and the well of the Transport System. Periodically, with a given frequency f_A, a certain number of pedestrians *"is created"* in A or, equivalently, "arrives there" from the exterior of C_ζ. This choice allows us to introduce an appropriate "placement function" $\varphi_P(t)$of the pedestrians in A. We will describe a possible $\varphi_P(t)$in section 4.

Suddenly after a time t_V , which is the time required to each pedestrian "to visit" a point P_i, pedestrians are asked to move immediately from P_i . It is necessary they move toward the next point P_{i+1} in a short time t_ϵ. In particular they must necessarily leave point P_i in a given short time t_ϵ. If in the time interval $(t_V\ , t_V + t_\epsilon)$ no buses will come to P_i , the aforementioned visitors will move by themselves. In this sense the probability $\pi(t - t_V)$ they leave P_i is

$$\{\pi(t - t_V) = 0, \forall t \le t_V\} \cup \{\pi(t - t_V) = 1, \forall t \ge t_V + t_\epsilon\}. \qquad (2)$$

This kind of diffusion is in competition with the Transport System. It is growing in time, it starts at t_V and becomes fully efficient at t_V+t_ϵ. This diffusion will thus be driven by the probability $\pi(t - t_V)$. In what follows we will adopt the simplified description:

$$\{\pi(t - t_V) = 1, \quad iff\ t = t_V + t_\epsilon\} \cup \{\pi(t - t_V) = 0, \forall t \ne t_V + t_\epsilon\}. \qquad (3)$$

We will moreover concentrate on a transport process having time duration of *one working day* T_d, i.e. about$8 - 10$ hours . We may thus write $t_0 \le t \le T_d$.

Located along the circuit, in points identified by the angular coordinate $\theta_i(t_0)$a certain number $i = 1, \ldots, N_B$of buses is ready to move. At the start buses are empty and in charge of picking up visitors to carry them along the circuit C_ζ. When a bus arrives at a stop P_i , some visitors will first get off the bus, suddenly other visitors will get on. The order of these operations indicates that the descent from the bus takes precedence over the ascent: this order will become important under certain conditions.

While $N_B$is constant in our model, the average number N_P(t) of the pedestrians conveyed at time $t$depends on the function$\varphi_P(t)$ computed in $[0, t]$. Unless $\varphi_P(t)$is a function with small amplitude and little frequency, it is to suppose N_P(t)will not decrease in the course of the day. This fact requires continuous adjustments on the buses frequency and speed. As previously said, the speed of the buses $V(t)$ is then externally regulated at equal time intervals, T, by the CS. This regulation operates in the same time on every bus present inC_ζ. Because we describe the process in terms of a circular uniform motion it seems safe to introduce the angular speed $\Omega(t)$ instead of the linear one $V(t)$.

The goal of this "sort of game" in the following will be to take in balance a variable set of visitors leaving *A* at time *t* with the number of visitors coming back to *A* later on by means of the lowest possible number N_B of buses and the best initial positioning of them. Other possible objectives may be trying to do the same also at point *B* and/or *C* and/or *D*.

3. Model development

As previously quoted, lot of methods have been developed for bus arrival time predictions. Among them let us just recall time series, artificial neural networks, Kalman filtering, digital simulation and Monte Carlo analysis. In these studies the recourse to discrete time sequences is the rule and generally successfully. Although we also face with discrete objects here, whenever possible we will try to describe the evolution process by means of continuous C_1 $(\mathbf{T}^1)$ functions, where $\boldsymbol{T}^1$ is the 1-dim torus on $\boldsymbol{R}^2$. So let us introduce the shortand notation $N(\theta, t)$ for the pedestrian occupation function $\sum_i (N(\theta, t)\, \delta(\theta - \theta_i))$, where the $\delta(x)$ is the Dirac's distribution and $i = A, B, C, D$.

Let us also remember we are assuming here the point of view of the Transport System benefit.

In other words, the function $N(\theta, t)$ of the pedestrians lying in $P(\theta, t)$ will be essentially different from zero only at point $\theta = \theta_i$. Quite generally we write

$$\mathrm{N}(\theta, \mathrm{t}_2) = \mathrm{N}(\theta, \mathrm{t}_1) +$$

$$+\int_{\mathrm{t}_1}^{\mathrm{t}_2} [\Delta \mathrm{N}(\theta.\mathrm{t}') - \Delta^* \mathrm{N}(\theta, \mathrm{t}')]\, \mathrm{dt}' - \Delta \mathrm{N}_\mathrm{D}(\vartheta, \mathrm{t}_1, \mathrm{t}_2) + \Delta^* \mathrm{N}_\mathrm{D}(\theta, \mathrm{t}_1, \mathrm{t}_2) \qquad (4)$$

The right hand side of eq.(4) corresponds to the sum of the initial number of pedestrians at $t = t_1$plus contributions. The term ΔN corresponds to the number of pedestrians who arrive by bus at $P(\theta)$ at time t'*and* visit $P(\theta)$; the term $\Delta^* N$gives the number of pedestrians leaving by bus $P(\theta)$at the time t'after visiting it; the term ΔN_D represents the number of pedestrians who leave *P* by diffusion between t_1 and t_2 and the term $\Delta^* N_D$ gives the number of pedestrians arriving at *P* by diffusion between t_1 and t_2.

As we said, diffusion from P_i arises when pedestrians urged to move does not match any bus in a short time. Thus pedestrians start to move with a speed $\omega_P \ll \Omega$ which is approximately constant. We interpret $\omega_P$as the mean to realize particle diffusionand the whole phenomenon analogous to brownian motion or 1-dim drift propagation[14]. Pedestrians start to diffuse and follow the stream, directed toP_j, , $j > i$. The primary goal of them is to reach point P_{i+1}. But this may happen only ifP_{i+1} is found to be not completely occupied. The spatial concentration of these pedestrians, which corresponds to our ΔN_D, is described in first approximation by Fick's theory. With the recourse to the familiar Propagator's expression for delta - shaped sources we obtain

$$\Delta N_D(\theta, t_1, t_2) = \int_\theta^{2\pi} d\theta' \int_{t_1}^{t_2} dt' \frac{A(\theta)}{\sqrt{4\alpha^2 \pi t'}} e^{-(\theta' - \theta)^2/(4\alpha^2 t')} \qquad (5)$$

An analogous result holds for the $\Delta^* N_D$ term, namely

$$\Delta^* N_D(\theta, t_1, t_2) = \int_0^\theta d\theta' \int_{t_1}^{t_2} dt' \frac{A(\theta')}{\sqrt{4\alpha^2 \pi t'}} e^{-(\theta' - \theta)^2/(4\alpha^2 t')} \qquad (6)$$

where $A(\theta)$ is the "initial concentration" of pedestrians located in $P(\theta)$ at $t = t_1$ and $D(t') \equiv \alpha^2 t'$ is a measure of average dispersion (or standard deviation) of diffusion at time t'. We will come back to that later on. Concerning the effect of diffusion we notice that the flux intensity is limited, being proportional to ΔN_D times the pedestrians speed, compared to the flux driven by bus, which is proportional to ΔN times Ω; nevertheless it is not completely negligible because diffusion starts much more frequently with respect to the average frequency of the bus stops.

An important observation concerning eq.(4) is that the θ-coordinate appearing in the ΔN and $\Delta^* N$ terms is not an independent variable, because it is related to the motion of the conveyors. In fact visitors who arrive at $P(\theta_i)$ at time *t* by means of buses have been necessarily at $P(\theta_{i-1})$ at the earlier time $t - (\theta_i - \theta_{i-1})/\Omega$. In other words the number of pedestrians $\Delta N(\theta_i, t)$ is *strictly connected* to the number of pedestrians $\Delta^* N(\theta_{i-1},\ t - (\theta_i - \theta_{i-1})/\Omega)$ and we can write

$$\theta(t_2) = \theta(t_1) + \Omega(t_1)(t_2 - t_1) \tag{7}$$

which holds for $t_2 - t_1$ smaller than one lap time *T*. The buses' capacity $Q(\theta(t), t)$, i.e. the number of seats free at time *t* and position θalso varies with time and stops. In this case the functional dependence $\theta = \theta(t)$ s absolutely necessary to describe the buses position. Starting from an assigned $Q(\theta_0, t_0)$, its evolution is regulated by the sum of the pedestrians continuously getting off and on at each bus stop. As quoted above we defined Q_M the maximum capacity of each bus, so that we can see $Q(\theta_0, t_0)$ as the sum of a set of Dirac's delta distributions centered at the buses starting points and having amplitude Q_M.
It is therefore

$$Q(\theta(t_2), t_2) = Q(\theta(t_1), t_1) - \int_{t_1}^{t_2} \left[\Delta N(\theta(t'), t') - \Delta^* N\big(\theta(t'), t^1\big)\right] dt' \tag{8}$$

The Transport System path is therefore a circular almost uniform collective motion of buses with short stops. Although stops occur at different times for different buses thus inducing modifications on the geometric structure of the system, on average, namely after a complete lap of the circuit, the structure will maintain its initial shape. The hypothesis that the visit times at strategic points remain exactly constant and the same everywhere hints to forecast a constant time period *T* between subsequent bus passes from the same points. This is good from the travelers' point of view but also for the conveyors. So we can speak of a *1-lap-averaged* angular speed $< \Omega >$ which is defined as

$$< \Omega >= \frac{2\pi}{T} \tag{9}$$

where we put

$$T = \frac{\theta_A - \theta_0}{\Omega} + \delta t_A + \frac{\theta_B - \theta_A}{\Omega} + \delta t_B + \frac{\theta_C - \theta_B}{\Omega} + \delta t_C + \frac{\theta_D - \theta_C}{\Omega} + \delta t_D + \frac{\theta_0 + 2\pi - \theta_D}{\Omega}\ . \tag{10}$$

Here δt_i , $i = 1, \dots, 4$ represents *the longest*, among all the buses in service, waiting time required by the visitors to get off or get on their bus at the stop P_i . Buses are supposed to be all the same capacity. If ϵ is the time required for a single pedestrian to leave the bus or remain on its place, let us set

$$\delta t_i = \varepsilon(\Delta^{\blacksquare} N_i + \Delta^{*\blacksquare} N_i)\left(1 + \gamma \frac{Q_M - Q_i}{Q_M}\right), \quad i = 1,\dots,4\ , \tag{11}$$

with

$$\Delta^{\blacksquare} N_i = max\{\Delta N_k(t_i)\},\ \ k = 1, \dots, N_B \quad \text{and analogously for } \Delta^{*\blacksquare} N_i \tag{12}$$

In eq.(11) we inserted a slight "time increase" factor $1 + \gamma\,(Q_M - Q_i)/Q_M$ into the bare scaling of ε by the factors $\Delta^{\blacksquare} N_i$, $\Delta^{*\blacksquare} N_i$. This effect in our opinion is induced by the greater or lesser crowding of the means of tran sport. Clearly eq.(10) holds only if $Q_i \leq Q_M$.

Collecting together eqs.(10), (11), (12), the $<\Omega>$ is related to the effective Ω by:

$$<\Omega> = \frac{2\pi}{\frac{2\pi}{\Omega} + \varepsilon \sum_j (\Delta^{\blacksquare} N_j + \Delta^{*\blacksquare} N_j)\left(1 + \gamma\left(Q_M - Q_j\right)/Q_M\right)}, \tag{13}$$

with $j = A, \ldots, D$. Notice however that in general $T = T(t)$.

In the next lap, whose time duration is by definition equal to *T*, the sites occupation numbers N_i will now become $N_i + \delta N_i$ so as the corresponding bus capacities$Q_i \equiv Q(\theta_i)$ will change to $Q_i + \delta Q_i$ **.** Hence we require to modify the angular speed Ω_t in the next lap to Ω_{t+T} ,

$$\Omega_{t+T} = \Omega_t + \delta\Omega_t \equiv \Omega' \tag{14}$$

under the condition

$$\frac{2\pi}{\Omega'} + \epsilon \sum_j (\Delta^{\blacksquare} N'_j + \Delta^{*\blacksquare} N'_j)(1 + \gamma \frac{Q_M - Q_j}{Q_M}) = \frac{2\pi}{\Omega} + \epsilon \sum_j (\Delta^{\blacksquare} N_j + \Delta^{*\blacksquare} N_j)(1 + \gamma \frac{Q_M - Q_j}{Q_M}) \cdot (1 + \gamma \frac{Q_M - Q_j}{Q_M}) , \tag{15}$$

where we defined $\Delta N'_j = \Delta N_j(t+T)$ and $\Delta^* N'_j = \Delta^* N_j(t+T)$. In this way we obtain a requirement on $<\Omega>$ such that the transport speed does not get worse while going from *t* to *t+T*, and *T remains constant.* All that gives

$$\delta\Omega_t = \Omega_t \frac{\delta R}{1 - \delta R} \tag{16}$$

having put

$$\delta R \equiv R(t) - R(t-T) \tag{17}$$

and where we introduced the ratio

$$R(t) = \frac{\epsilon\Omega}{2\pi} \sum_j (\Delta^{\blacksquare} N_j + \Delta^{*\blacksquare} N_j)(1 + \gamma \frac{Q_M - Q_j}{Q_M}) \tag{18}$$

with $t \in [(k-1)T, kT]$ and $k \in \boldsymbol{N}$.

The $R(t)$ is in fact the ratio between the fraction of the time period *T* when buses stop at $P(\theta_i)$, *i = 1, ...,4*, and the fraction in which buses move around C_ζ. If this ratio does not change between $t - T$ and *t* , Ω_t does not change in going from *t* to $t + T$. We are also supposing $\epsilon\Omega$ small enough to have $\delta R \ll 1$; this is so because by assumption the motion is nearly circular uniform with short bus stops. The structure of eq.(16) is such that to enhance the positive δR more than the negative ones. In this way, it is encouraged a situation in which the overall stop time at bus stops during the laps is getting smaller, while the opposite situation tends to stabilize.

Visitor stopover at strategic points takes place according to the modalities prescribed in the ΔN, $\Delta^* N$, ΔN_D and $\Delta^* N_D$ definitions. To this aim we need to distinguish between $N^\wedge(\theta, t)$, which is the number of *pedestrians ready to move* from θat time *t*, $N(\theta, t)$ that is the number of pedestrians present in θ at time *t* and $N^*(\theta, t)$ namely the number of pedestrians actually transferred with bus from θ at time *t*. The time periodic structure of the process, in which we put *T* unchanged all the day suggests the adoption of the variable

$$t_{(k)} \equiv t - [t/T]T = t - kT \tag{19}$$

with $k \equiv [t/T] \in \boldsymbol{N}$, instead of the simple $t \in \boldsymbol{R}$**.** Il is then

$$0 \leq t_{(k)} \leq T , \quad k \in \boldsymbol{N} . \tag{20}$$

Under the hypothesis $t_V < T$ we thus obtain

$$N^{\wedge}(\theta,\ t_{(k)}) = \Delta N(\theta, t_{(k)} - t_V) + \Delta N_D(\theta, t_{(k)} - t_V) = \tag{21}$$

$$= \int_0^{\theta} N^* \left(\theta', \mathrm{t_{(k)}} - \frac{\theta - \theta'}{\Omega_k} - \mathrm{t_V}\right) d\theta' \tag{22}$$

$$N^*(\theta, t_{(k)}) = min\left\{N^{\wedge}\left(\theta, t_{(k)}\right),\ Q(\theta_0, t_{(k)} - \tfrac{\theta - \theta_0}{\Omega_k})\ \right\} \tag{23}$$

where $\theta' < \theta, t_{(k)} \in\ [0, T),\ \ \Omega_k \equiv \Omega_{t_{(k)}}$.

In eq.(23) we underlined the need of an adequate matching between the number of pedestrians ready to move, the number of seats free and the circular motion of the "seat carriers".

Two additional elements we included in our development. The first one is the fact that in the course of the day visitors moving from *A* may choose to skip visit *B* and go directly to visit *C* if the time at their disposal until the end of the working day T_D is poor. As the hours go on, this trend will extend to points *C* and *D* too. As a consequence the preferred direction starting from *A* will change in the course of the day. Approximately this will result in a continuous slight change of visitors' distribution along the destinations. Quite similar considerations hold even for the arrivals to *C* and *D* starting from *B* and finally from *C*. with the passing of the hours. The second element is the fact that sites *A, B, C, D* do not have infinite capacity. If a site is getting full of visitors less people will be interested to visit it in their immediate future. In fact if a site $P(\theta_i)$is full at time *t, visiting it becomes forbidden* until is $N(\theta_i, t) = C_i$, if we defined C_i the capacity of the sites *A, ..., D*.

Even from a psychological point of view, a pedestrian may privilege to visit sites not completely occupied, nor completely empty. The preferred situation would be to find a site just half occupied. It is possible to take into account the last two issues simply by introducing ad hoc two multiplicative factors. We call them $\mathfrak{D}_i(\theta, t)$ and $\mathfrak{F}(\theta, t)$. The first one is the probability of moving toward θ starting from θ_i and then can be called the "*preferred Destination*" function $\mathfrak{D}_i(\theta, t)$. The second one is an indication of the "*Filling status*" of a site θ. In practice both functions $\mathfrak{D}_i(\theta, t)$ and $\mathfrak{F}(\theta, t)$ take non zero values only at $\theta = \theta_i$.

These functions, moreover, are two different filters acting on the pedestrians' destination. As such, both $\mathfrak{D}_i(\theta, t)$and $\mathfrak{F}(\theta, t)$ take values ranging from zero to one. In the next section a specific realization of them will be exhibited.

With the inclusion of these additional factors we modify

$$N^* \rightarrow\ N^* \mathfrak{D}_i \mathfrak{F} \tag{24}$$

In the light of all considerations made, we rewrite eq.(4) as

$$N(\theta, t_0 + kT \leq t < (k+1)T)\ \equiv N\left(\theta, t_{(k)}\right) =$$

$$= N\left(\theta, t_{(k-1)}\right) + \int_0^T [\Delta N_k(\theta, t') - \Delta^* N_k(\theta, t')]dt' - \Delta N_D\ + \Delta^* N_D\left(\theta, t_{(k)}\right) \tag{25}$$

where we defined

$$\Delta N_k(\theta, t') \equiv\ \Delta N\left(\theta, t'_{(k)}\right) \tag{26}$$

with the additional condition $N\left(\theta, t'_{(-1)}\right) \equiv 0$. In eq.(25) we still have to substitute in the $\Delta -$ and $\Delta^* -$ terms the expressions found in accordante with eqs. (3), (21), (23) and (24). We obtain this way

$$\Delta N_k(\theta, t) =\ \mathfrak{D}_k(\theta, t)\mathfrak{F}(\theta, t) \int_0^{t_\varepsilon} N^*\left(\theta_0,\ t_{(k)} - \tfrac{\theta - \theta_0}{\Omega_k} - t_V - t'\right) \cdot \left(1 - \pi(t')\right)dt', \tag{27}$$

$$\Delta^* N_k(\theta,t) = \int_0^{t_\varepsilon} N^*(\theta, t_{(k)} - t_V - t')(1 - \pi(t')dt' \qquad (28)$$

and correspondingly the eqs. (5), (6) are now changed to

$$\Delta N_{k,D}(\theta,t) = \int_\theta^{2\pi} d\theta' \int_{t_0}^{T} dt' A(t_{(k)})\, e^{-(\theta'-\theta)^2/(4\alpha^2 t')} \qquad (29)$$

and

$$\Delta^* N_{k,D}(\theta,t) = \int_0^{\theta} d\theta' \int_{t_0}^{T} dt' A(t'_{(k)}) \qquad (30)$$

having put

$$A\big(t_{(k)}\big) = \frac{max\{\,0\,,\ N(\theta\,,t_{(k)} - t_V - t_\varepsilon) - \Delta^* N(\theta\,,\ t_{(k)} - t_V - t_\varepsilon) - Q(\theta\,,\ t_{(k)})\}}{\sqrt{4\alpha^2 \pi t_{(k)}}} \qquad (31)$$

As known the α-coefficient appearing in eqs. (29),(30),(31), is the diffusion coefficient or *diffusivity,* introduced in the treatment of thermal diffusion, as well as matter, momentum or pressure diffusion. In these cases the α- coefficient is strongly dependent on the local framework in which diffusion occurs and gives a measurement of the rapidity of the collective motion. If ω_p is exactly constant the diffusivity becomes zero almost everywhere. In our case we observe, in passing, how *diffusion does not begins* at all even in the special situations *when*

$$t_V = m\,(\theta_i - \theta_{i-1})/\Omega_k \ , \qquad i \in \boldsymbol{N}/(\boldsymbol{4}) \ , \qquad m \in \boldsymbol{N} \qquad (32)$$

for $t \in \big(t_{(k)}\,, t_{(k+1)}\big]$.

Concerning the diffusion coefficient α, let us consider the Einstein - Smoluchowski equation[**15**] and applications. As we can see from general treatments, α^2 is essentially determined by the mobility parameter $\mu$of the diffusing medium multiplied by the ratio $\rho/(\partial\rho/\partial U)$, where ρ is the number density of diffusing particles and U il the potential of the "force" that creates the drift motion. When the distribution of ρ is described by classical Maxwell - Boltzmann statistics, the α^2 most general expression reduces to $\alpha^2 = -\mu k_B T$ where k_B is the Boltzmann constant, T the absolute temperature. This last result shows in dimensional terms how diffusivity is given by the product of a mobility parameter times the average energy density for single diffusing particle. In our case the mobility parameter is expressed by the speed $\omega_p$whereas the average energy per particle is constant in first approximation(it may change slightly with time and density). This leads us to write

$$\alpha^2 \equiv K\omega_p \qquad (33)$$

The equations (4) and (8) previously discussed and modified under the prescriptions done in eqs.(25)-(31), are the required system of integral Fredholm equations with iterative core. We are interested in its Diophantine solutions.

The free parameters we can modify in order to improve the service of this transport system reduce essentially to three, i.e. the number of buses employed, the initial angular positioning of buses and the angular speed changes. The good quality of the service can be evaluated in the aftermath by three parameters stemming after one day activity. They are the ratio $\langle PL\rangle$ of the total number of pedestrians conveyed with respect to the total length traveled by the buses at work *(pedestrians conveyed per unit length)*; the average capacity $\langle AC\rangle$ of the buses used *(optimal bus capacity)*; the ratio $\langle BS\rangle$ of the total number of pedestrians lost by diffusion with the total number of pedestrians arrived at point *A* by bus *(best bus synchronization).*

Improving the service means possibly optimize these parameters. They are written as:

$$\langle PL\rangle \equiv \sum_{i=1}^{i_m} \int_0^T \sum_{k=A}^{D} \Delta N\big(\theta_k\,, t'_{(i)}\big)\, dt'_{(i)} / N_B \sum_{i=0}^{i_m} \Omega_i T \qquad (34)$$

$$\langle AC \rangle \equiv \frac{1}{i_m T}\sum_{i=1}^{i_m}\frac{1}{Q_M N_B}\int_0^T \sum_{k=A}^{D} Q(\theta_k, t'_{(i)})\ dt'_{(i)} \tag{35}$$

$$\langle BS \rangle \equiv \sum_{i=1}^{i_m}\int_0^T \sum_{k=A}^{D}\Delta N_D\,(\theta_k, t'_{(i)})\ dt'_{(i)} \Big/ \int_0^T \varphi_P(t')dt' \tag{36}$$

where $i_m = [t/T]$, and $0 \le t'_{(i)} \le T$.

At least one other question, among others, still needs to be raised. That is: what happens to a visitor who arrives at site *D* without stopping first at all previous points? Can he repeat the round again, to stop at the missing points or not? Answering affirmatively or not will lead to different optimization schemes and different problems to be solved.In the next section we will look at one.

4. A first realization

Let us now put at work this model and examine the performance in a specific situation. It matter of a very simple configuration. We introduce first the following description of the placement function $\varphi_P(t)$, useful for applications. It is the sum of a set of equally spaced "solitons-like" profiles (the pedestrians) who arrive continuously to point *A* at regular time intervals $\delta t = 1/f_a$. It is then

$$\varphi_P(\theta, t) = \sum_{j=0}^{j_M} A_0 sech\left[\epsilon_0\left(\theta - f_a\left(t - t_j\right)\right)\right]\delta(\theta - \theta_0) \tag{37}$$

where $\delta(x)$, i.e. the Dirac delta distribution, is needed to stop the arrivals at site *A*. The initial distribution of the buses is analogously described, we suppose, by means of a small set of "soliton-like" profiles. Buses are spatially distributed at the beginning of the working day and left to move subsequently. The spatial structure of the train of buses remains therefore unchanged.

We write this as

$$Q(\theta, t) = \sum_{k=1}^{N_B} b(\theta_k, t)\, q_k(\theta - \Omega t) \tag{38}$$

$$b(\theta_k, t) = b(\theta_k, t_0) - \int_0^t [\,\Delta N(\theta_k(t'), t') - \Delta^* N(\theta_k(t'), t')\,]dt' \tag{39}$$

$$q_k(\theta - \Omega t) \equiv sech[\eta\,(\theta - \theta_k - \Omega t)] \tag{40}$$

A possible realization of the so called "preferred Destination" functions $\mathfrak{D}_i(\theta_j, t)$ previously introduced with relation (24) may be given by

$$\sum_j \mathfrak{D}_1(\theta_j, x) = \mathfrak{D}_1(\theta_2, x) + \mathfrak{D}_1(\theta_3, x) + \mathfrak{D}_1(\theta_4, x) = $$
$$= \frac{a_{12}\sqrt{x}}{1 + b_{12}x} + \frac{a_{13}}{1 + b_{13}\,(0.5 - x)^2} + \frac{a_{14}\sqrt{1-x}}{1 + b_{14}(1-x)} \tag{41}$$

$$\sum_j \mathfrak{D}_2(\theta_j, x) = \mathfrak{D}_2(\theta_3, x) + \mathfrak{D}_1(\theta_4, x) = $$
$$= \frac{a_{23}\sqrt{x}}{1 + b_{23}\,x} + \frac{a_{24}\sqrt{1-x}}{1 + b_{24}(1-x)} \tag{42}$$

$$\mathfrak{D}_3(\theta_4, x) = \frac{a_{34}\sqrt{x}}{1 + b_{34}\,x} \tag{43}$$

having set $x \equiv (t/T_D) \le 1$. By putting then $i = 1,2,3,4 \equiv A, B, C, D$ we can obviously write

$$j = i + k\,, \quad k = 1,2,3\,, \quad |j| \le 4 \tag{44}$$

Equations (41)-(43) require a normalizationN_m ,such that

$$(1/N_m)\sum_j \mathfrak{D}_i(\theta_j\,, x) = 1\,, \qquad 0 < x \leq 1\,, \qquad\qquad i = 1{,}2\,. \tag{45}$$

In fact, only the function $\mathfrak{D}_3(\theta_4\,, x)$ can remain less than one, if we are at t near T_D, because visitors at point *C* may also prefer to stay still and visit it calmly instead of moving.

With respect to the so called "Filling status" function $\mathfrak{F}(\theta_i, t)$ we supposed it has the structure:

$$\mathfrak{F}(y_i) = \frac{1}{2} + 2y_i - 2y_i^2 \tag{46}$$

where it isdefined $y_i = N(\theta_i\,, t)/C_M$, and the capacities of the sites are given by $C_i \equiv C_M = 220$, for $i = A, B, C, D$.

The three different contributions to the function $\sum_j \mathfrak{D}_1(\theta_j, x)$ introduced with the eq.(41) are graphically represented in the *Figure 1*.

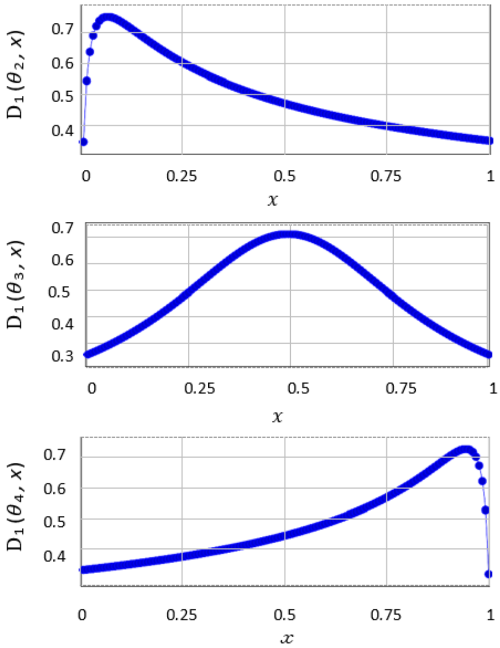


Figure 1: The behavior with respect to the time of the so called "preferred Destination" function for pedestrians starting from A and moving, respectively, toward points B, C and D.

Other necessary considerations include to relate the "time width" ϵ_0 of the single ‘soliton-like’ function, to the pedestrians arrival frequencyf_a and to evaluate the t_V and T_D magnitudes. To fix ideas let us assume a *time unit*t_u and duration T_D

$$1t_u \equiv 5\, min, \qquad\qquad T_D = 120\; t_u \tag{47}$$

and, consequently, in t_u units,

$$t_\varepsilon = 2\,, \qquad f_a \cong 2\,, \qquad \eta = 4\,, \qquad t_V = 12 \tag{48}$$

As for as the δt_i ‘*crowding-induced’* time delay introduced in eq.(11) and other parameters, occurring in the model, we will assume here the following values

$$\varepsilon \approx 0.01\,, \qquad \gamma = 0.5\,, \qquad K \approx 3\,, \qquad \omega_p = \Omega_0/12 \tag{49}$$

that gives

$$\delta t = 1.5625\,, \qquad \alpha = 0.1256625 \tag{50}$$

Computer graphic realizations concerned with the function$\varphi_P(\theta_0\ ,t)$ presented in eq.(38) show a pattern like that in *Figure2*. We supposed there A_0=8:

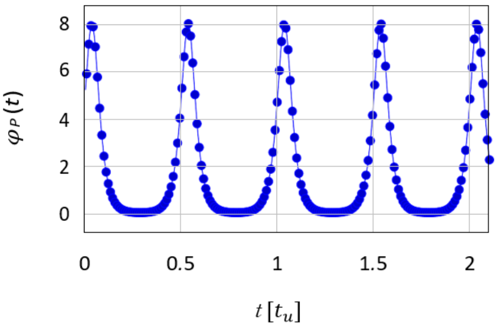


Figure 2: The $\varphi_P(t)$ "placement function" of the pedestrians at site A.

As for as the angular velocity at the beginning,Ω_0, we assumed the following:

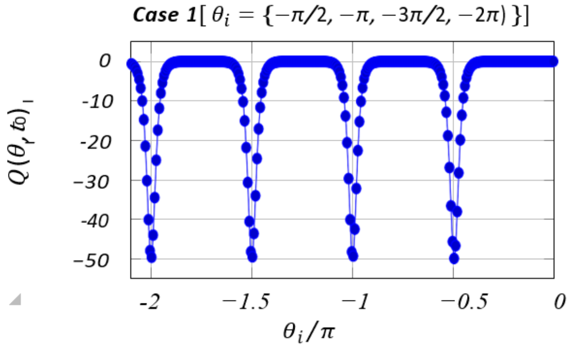
$$\int_0^{2\pi/\Omega_0} \varphi_P(t')dt' = \int_0^{2\pi} Q(\theta',t_0)d\theta' \ , \tag{51}$$

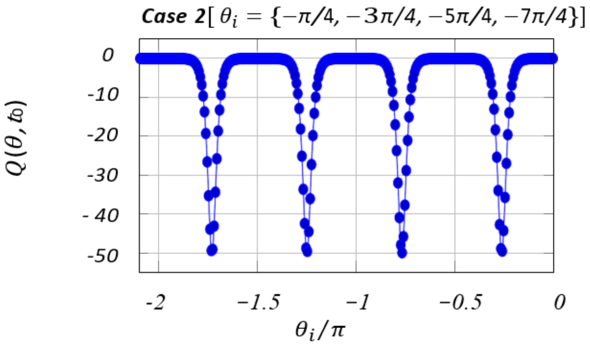
to get

$$\Omega_0 = 0.50265\,, \qquad\qquad T = 12.5\ . \tag{52}$$

Making use of the explicit expression done in eqs. (38)-(40), we find analogous computer simulations for $Q(\theta,t)$.

Most in particular we analyzed the evolution of the system starting from three different initial *angular positioning distributions* of the buses at the time $t_0 = 0$. We will also suppose we have *four buses* moving counterclockwise and put $Q_M = 50$. The initial bus placements are shown in the *Figure 3*.





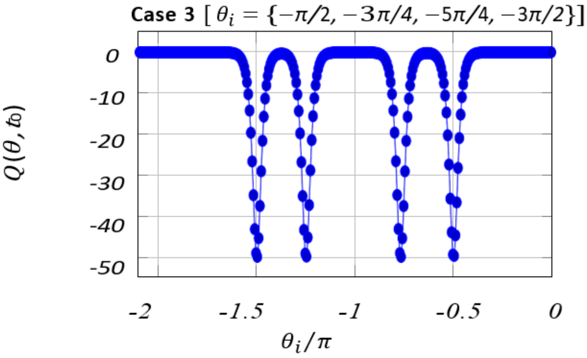


Figure 3: The three sets of bus initial placements on the circuit C_ζ considered in our simulation

The sites of interest, corresponding also to the bus stops, i.e. the points *A,B,C,D,* are located, respectively, at the angular coordinates $\theta = \{0\,,\pi/2\,,\pi\,,3\pi/2\}$, as is shown in the following *Figure 4*.

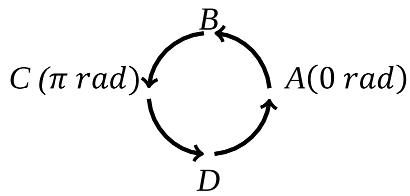


Figure 4: The circuit C_ζ and its interest points *A*, *B*, *C*, *D*.

Concerning the final question raised at the end of the previous section, in this case we supposed pedestrians are allowed to repeat the round only if the sites they have not yet visited are beyond the site of "greatest interest" at that time, identified by the functions $\mathfrak{D}_i(\theta_j\,,t)$ shown in Fig.1. Checking now the performance parameters (see eqs.(34)-(36)) measured at times $t_k = t_0 + kT\,,\ k = 5{,}10{,}15{,}20$, for the **Cases1,2,3** of before, gives us an estimate of the goodness of the transport system model and the particular configurations examined. We obtained the following results, shown in Tables 1, 2, 3.

Table 1: Case 1

k (ordinal n of the lap)	$\langle PL \rangle$	$\langle AC \rangle$	$\langle BS \rangle$
05	9.73	0.18	0.18
10	9.23	0.12	0.21
15	8.39	0.09	0.25
20	7.45	0.08	0.28

Table 2: Case 2

k (ordinal n of the lap)	$\langle PL \rangle$	$\langle AC \rangle$	$\langle BS \rangle$
05	8.08	0.24	0.12
10	7.51	0.20	0.20
15	7.16	0.12	0.24
20	6.97	0.10	0.30

Table 3: Case 3

k (ordinal n of the lap)	$\langle PL \rangle$	$\langle AC \rangle$	$\langle BS \rangle$
05	8.98	0.22	0.19
10	8.52	0.17	0.21
15	8.20	0.11	0.24
20	7.89	0.08	0.27

Remembering that the goal of the service is to maximize the first parameter while minimizing the other two, the results shown in Tables 1,2,3, even though preliminary, suggest what follows. The number of the pedestrians conveyed per unit length in the first fraction of the time is about *8.5*but it decreases in time, with a rate about $0.1/T$. The fraction of the number of seats not occupied on the buses with respect to the allowable one is initially about *20%* and it decreases in time with a rate about $1.2\%/T$. The number of the pedestrians "lost by the buses", because they gone by feet, with respect to the total number of pedestrians entered C_ζ, varies also with time, ranging from ~ 0.17 at the beginning of the service until ~ 0.28after about *100* minutes. Moreover, an initial displacement of the buses not perfectly symmetric with respect to the circuit $C_\zeta$seems to perform slightly better than a symmetrical placement. We are aware, however, that a longer observation time and related simulation would be needed.

5. Conclusion

We developed the study of a quite generic linear configuration in transport system optimization problems and examined some tentative solutions. The problem proposed is essentially a theoretical exercise of transport kinematics with the inclusion of parameters coming from statistical data. These last include preferences of the visitors with respect to the sites to be visited, a psychological rejection by visitors themselves of crowded or desert places and, also, the more or less patience in waiting for the means of transport. The weight of these parameters in the development of the model may look of marginal relevance at the beginning, but it subsequently becomes noticeable because little changes in these parameters give rise to a rich variety of evolutions. Although this analysis is a theoretical one, the proposed treatment can be extended to practical situations simply by adding the right sets of values to the parameters included in the model.

In the case at hand four buses serve an ever growing set of visitors to reach four strategic points. The goal was trying to minimize dispersion of visitors from the given points meanwhile minimize the time to complete the visits of visitors too and to avoid to have a service with low bus occupancies.

From the three different configurations examined here, referring to four equally spaced visit points, we observed through simulations as a spatially symmetric buses distribution is a bit more efficient than others only in the first time period of the service. On the contrary, in the remaining time period, preliminary simulations indicate as a slight spatial asymmetry of the initial positioning satisfies better the requirements for further optimization. Time asymmetries in the arrival of the buses at the visit points become useful, indeed, in reducing diffusion effects of the pedestrians and overpopulation of the sites. Needless to say, the observations just done hold only for our case of full symmetries, concerning then buses, sites, capacities, velocities, frequency of arrivals and so on. It would be more interesting to try to apply the theory here presented in other, more realistic situations. The author will be responsible for making this extension in the near future.

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