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Interpolated Coefficients Finite Element Method for Nonlinear Two-Point Boundary Value Problems

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Abstract: In this paper, we consider nonlinear two-point boundary value problem using the Interpolated Coefficients Finite Element Method (ICFEM). We use the slice k-degree polynomial interpolation for nonlinear term and use Newton's method to solve the nonlinear equation system. We find the error convergence order of the ICFEM has some obvious characteristics. When k is an odd number, the error order is the normal finite element convergence order. And when k is an even number, the error convergence order has super convergence. The numerical results show the error convergence order of the 3rd ICFEM at the nodes is basically the normal finite element convergence order k + 1, and k = 2,4, the error convergence order of the 4th ICFEM at the node is generally higher than k + 1, almost reaching the 2k order super.

1. Introduction

In recent years, in the fields of science, technology and engineering, we are often faced with the problem of solving Nonlinear Two-Point Boundary Value Problems. For instance, to exploit resources like oil, natural gas, etc., we need study the problem of seepage in underground porous media; to predict weather conditions, we need solve the fluid dynamics and thermodynamic equations that describe atmospheric motion. The phenomenon described by singular perturbation problem is often singular in a local area, its solution contains a boundary layer or an inner layer, and the solution or its derivative changes very drastically in this area. Its release has relation to small perturbation parameters in addition to variables.

If a numerical method is used to solve Nonlinear Two-Point Boundary Value Problems on a uniform mesh, to achieve certain calculation accuracy, the local singularity will result in too fine mesh on the solution area, which will cause unnecessary calculation time and waste of data storage. Moreover, solution under uniform mesh will produce non-physical oscillation in the sharply changing area of the solution, leading to unsatisfactory results. Layer adaptive mesh is a non-uniform mesh that can effectively solve singular perturbation problems [1], including Shishkin mesh [2], Bakhvalov mesh [3,4], Bakhvalov-Shishkin mesh [5], etc. This type of mesh has local encryption in the boundary layer. Scholars have studied various singular perturbation problems on the layer adaptive mesh, and some research results have been achieved, such as literature [6-7]. In the construction of layer adaptive mesh, we need select mesh transition points to determine the mesh distribution. Therefore, how to select parameters in the mesh transition points so that the numerical solution can better approximate the exact solution is a very meaningful work. Literature solves a class of singular perturbation problems by using differential evolution algorithm to optimize parameters [8-9]. Literature uses differential evolution algorithm to solve a type of singular perturbation problems involving two parameters on Shishkin mesh.

Interpolated Coefficients Finite Element Method (ICFEM) finds the solution through iteration. At present, ICFEM has been widely used in linear Two-Point Boundary Value Problems.In this paper, we consider nonlinear two-point boundary value problem using the Interpolated Coefficients Finite Element Method (ICFEM).

2. Interpolated Coefficients Finite Element Method

In this work, we consider nonlinear two-point boundary value problem with k = 2, 3, 4 using the Interpolated Coefficients Finite Element Method (ICFEM).

$$\begin{cases} -u'' + u^3 = f, \\ u(0) = u'(1) = 0, \end{cases} \tag{1}$$

Where $x \in I := (0,1)$ and f is determined by the exact solution $u = \sin x(2-x)$.

Let $H^1(I) := \{u \in L^1_{loc}(I) : \|u\|_1 = \int_I (u^2 + (u')^2)dx < \infty\}$, $S_0 := \{u \in H^1(I) : u(0) = 0\}$, then the weak form of (1) is

$$\begin{cases} find\ u \in S_0\ such\ that\ \forall v \in S_{.0}, \\ (u', v') + (u^3, v) = (f, v), \end{cases} \tag{2}$$

For a given constant N, with step h = 1/N, $\tau^{\mathrm{h}} = \{x_j = jh, j = 0,1,\cdots,N\}$. A finite-dimensional subspace of S_0 is

$S_0^{\ k,h} := \{u \in S_0 : u\big|_{e_j} \in P^k(e_j), j = 1,2,\cdots,N\}$,

which is the k-order finite element space under the partition τ^h, where $P^k(e_j)$ represents polynomials set of degree less than k on the element $e_j = (x_{j-1}, x_j)$. Then the Ritz-Galerkin finite element approximation of (1) can be expressed as

$$\begin{cases} find\ u_h \in S_0^{\ k,h}\ such\ that\ \forall v_h \in S_0^{\ k,h}, \\ (u_h', v_h') + (u_h^{\ 3}, v_h) = (f, v_h). \end{cases} \tag{3}$$

It can be found that when solving the nonlinear system of equations (3) by the Newton method, the tangent matrix needs to be calculated many times, and the workload is very large. Next, consider the interpolation coefficient finite element method for solving the nonlinear two-point boundary value problem (1).

Suppose $\{\phi_i\}_{i=1}^d$ is the basis function of the k-order finite element, where $d = \dim S_0^{k,h}$. Let $I_h : S_0 \to S_0^{\ k,h}$ is a slice k-degree polynomial interpolation operator, then the finite element solution is $u_h = I_h u = \sum_{j=1}^d u_j \phi_j$. Now using the same slice k-degree polynomial interpolation for u^3, we have $I_h u^3 = \sum_{j=1}^d u_j^{\ 3} \phi_j$.

substitite for $u_h^{\,3}$ with above formula, we get

$$\begin{cases} find\ u_h \in S_0^{\,k,h}\ such\ that\ \forall v_h \in S_0^{\,k,h}, \\ (u_h', v_h') + (I_h(u^3), v_h) = (f, v_h). \end{cases} \tag{4}$$

Let $v = \phi_i, i = 1, 2, \cdots, d$ in (4), the nonlinear equation system as following

$$\sum_{j=1}^{d}((\phi_j', \phi_i')u_j + (\phi_j, \phi_i)u_j^3) = (f, \phi_j.), \qquad \forall i = 1, 2, \cdots, d. \tag{5}$$

Let

$$k_{ij} = (\phi_j', \phi_i'), \quad K = (k_{ij})_{1\le i,j\le d},$$

$$m_{ij} = (\phi_j, \phi_i), \quad M = (m_{ij})_{1\le i,j\le d},$$

$$b_i = (f, \phi_i), \quad B = (b_1, b_2, \cdots, b_d)^T,$$

$$U = (u_1, u_2, \cdots, u_d)^T, \quad F(U) = (u_1^{\,3}, u_2^{\,3}, \cdots, u_d^{\,3})^T,$$

Then (5) can be written in matrix form

$$G(U) = KU + M[F(U)] - B = 0. \tag{6}$$

Finally, the nonlinear equation system (6) is solved by Newton's method. Through direct calculation, the Frechet derivative of G (U) (the Jacobi matrix) can be obtained

$$G'(U) = K + M[F'(U)] = K + Mdiag(3u_1^2, \cdots, 3u_d^2)$$

$$= K + M\begin{pmatrix} 3u_1^2 & 0 & \cdots & 0 \\ 0 & 3u_2^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & 3u_d^2 \end{pmatrix}. \tag{7}$$

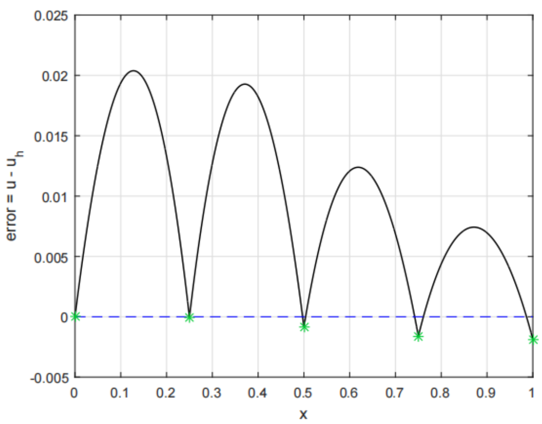
Then, for a given initial value U^0, the Newton iteration format for solving (6) is

$$U^{k+1} = U^k + \alpha_k V^k, \tag{8}$$

Where α_k is the step factor and V^k such that $G'(U^k)V^k = -G(U^k)$.

3. Results and Analysis

First, divide $I = (0,1)$ into 4 units, and use $k = 1, 2, 3$ and 4 Interpolation Coefficients Finite Element Method (ICFEM) to solve the nonlinear two-point boundary value problem (1), and the error curve is obtained as follows. As can be seen from the Figure 1-4, the accuracy is very high at the end nodes of the element (marked with * in the figure). For the k = 2, 4 (even) cases, there is also high precision at the midpoint of the cell (marked with + in the figure), while for k = 1, 3 (odd) cases, the error at the element midpoint is almost the largest on the entire element.



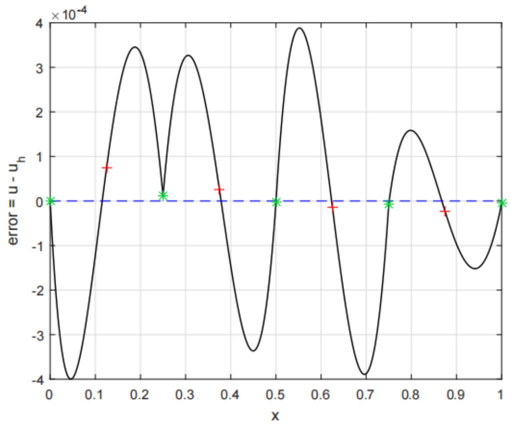
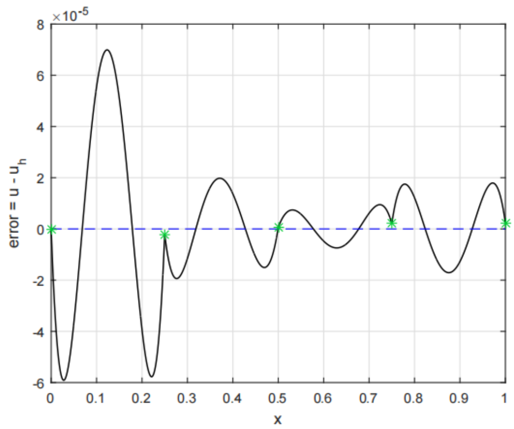


Figure 1: Error curves for ICFEM (k=1) Figure 2: Error curves for ICFEM (k=2)



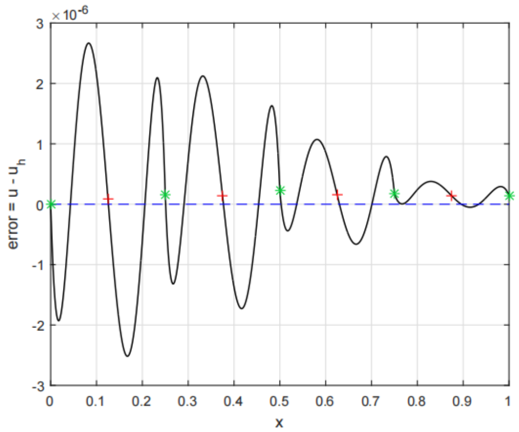


Figure 3: Error curves for ICFEM (k=3) Figure 4: Error curves for ICFEM (k=4)

Define discrete L^2 error as following

$$error_N = \left(\frac{1}{N} \sum_{j=0}^{N} (u(x_j) - (u_h)_j)^2 \right)^{1/2}, \tag{9}$$

Where $x_j (j = 0,1,\cdots,N)$ is the node of the element, $(u_h)_j$ is the value at node x_j of the ICFEM solution u_h .use $k = 2,3$ and 4 Interpolation Coefficients Finite Element Method (ICFEM) to solve (1) for N = 2, 4, 8, 16, 32, 64. Discrete L^2 error is shown in Table 1.

The "order" in the table refers to the error based on the two calculations before and after the estimated value of the convergence order obtained from the ratio of the differences, and its calculation formula is

$$\text{order}_{N_1,N_2} = \frac{\log(error_{N_1}/error_{N_2})}{\text{og}(N_1/N_2)}, \quad N_1 < N_2. \tag{10}$$

In Table 1, the number of units before and after is exactly 2 times the relationship, so the calculation of order is simplified to log2 (errorN / error2N). From the Table 1, it can be seen the solution of problem (1) for ICFEM (k=2) has the 4th-order convergence rate, the solution of problem (1) for ICFEM (k=4) has the 6th-order convergence rate. They have super-convergence.

While k = 3, the solution of problem (1) has ordinary finite element convergence rate of order 4.

Table 1: The discrete L^2 errors of k = 2, 3, and 4

N	k=2		k=3		k = 4	
	error	order	error	order	error	order
2	2.8909E-04	N/A	1.8485E-05	N/A	1.6821E-05	N/A
4	7.2206E-06	5.32	1.9046E-06	3.28	1.7469E-07	6.59
8	4.9480E-07	3.87	1.2041E-07	3.98	2.2349E-09	6.29
16	3.1205E-08	3.99	7.3961E-09	4.03	3.2744E-11	6.09
32	1.9530E-09	4.00	4.5707E-10	4.02	1.4574E-13	7.81
64	1.2206E-10	4.00	2.8255E-11	4.02	1.8423E-13	-

Further, we plot the discrete L^2 errors of k = 1, 2, 3, 4, and 5 times ICFEM under various divisions as shown in Figure 5. It can be found that the convergence speeds of ICFEM (k =2, 3) are almost the same, again showing that the ICFEM (k=2) has a 4th-order super convergence. Similarly, Figure 5 also shows that the 4th-order ICFEM's convergence speed is comparable to that of k = 5, and even exceeds the convergence speed of k = 5.

Finally, the errors for k = 1, 2, 3, 4 ICFEM of N = 4, 8, 16, 32 at the node are listed in Tables 2-5, respectively. The numbers in parentheses in brackets are the convergence order calculated from the ratio of the errors before and after, the calculation formula is shown in (10). From the data in the Table2-5, the error convergence order of the 3rd ICFEM at the nodes is basically the normal finite element convergence order k + 1, and k = 2, 4, the error convergence order of the 4th ICFEM at the node is generally higher than k + 1, almost reaching the 2k order super.

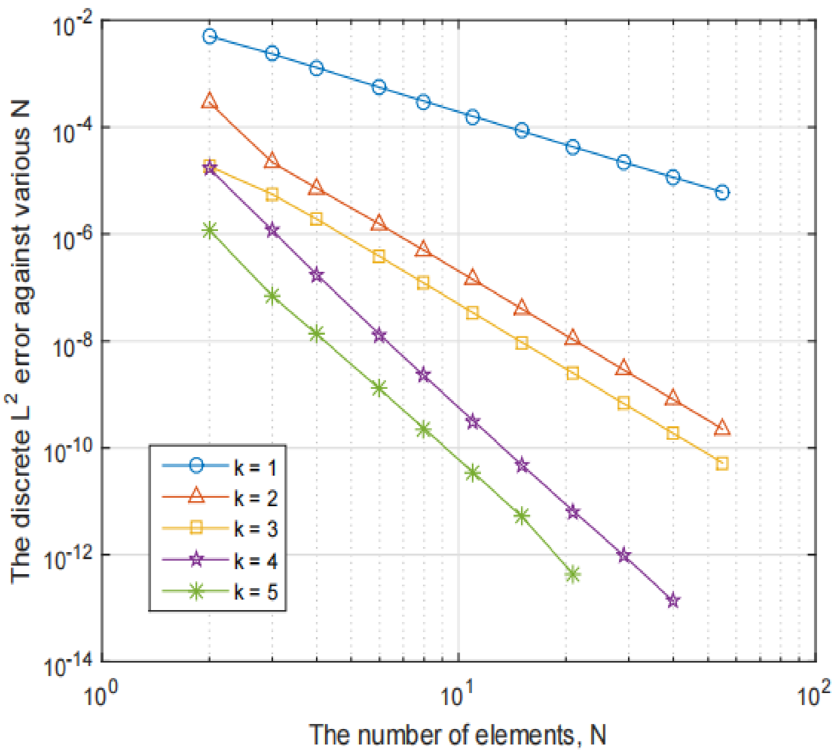


Figure 5: Error curves for ICFEM (k=1, 2, 3, 4, 5)

Table 2: Errors of ICFEM (k=1) at nodal points for N = 4, 8, 16, 32

N	4	8	16	32
x = 1/4	4.3556E-05	2.6471E-06(4.04)	6.0739E-08(5.45)	2.3282E-08(1.38)
x = 1/2	8.4443E-04	2.1454E-04(1.98)	5.3717E-05(2.00)	1.3433E-05(2.00)
x= 3/4	1.6015E-03	4.0540E-04(1.98)	1.0157E-04(2.00)	2.5403E-05(2.00)
x = 1	1.8623E-03	4.6942E-04(1.99)	1.1751E-04(2.00)	2.9386E-05(2.00)

Table 3: Errors of ICFEM (k=2) at nodal points for N = 4, 8, 16, 32

N	4	8	16	32
x = 1/4	1.1789E-05	8.6422E-07(3.77)	5.5416E-08(3.96)	3.4833E-09(3.99)
x = 1/2	3.4838E-06	8.6422E-07(3.77)	5.5416E-08(3.96)	3.4833E-09(3.99)
x= 3/4	6.0159E-06	2.3161E-07(4.70)	1.2539E-08(4.21)	7.5467E-10(4.05)
x = 1	4.6080E-06	1.6589E-07(4.80)	8.7559E-09(4.24)	5.2320E-10(4.06)

Table 4: Errors of ICFEM (k=3) at nodal points for N = 4, 8, 16, 32

N	4	8	16	32
x = 1/4	2.1011E-06	1.1910E-07(4.14)	7.3411E-09(4.02)	4.5724E-10(4.00)
x = 1/2	6.1175E-07	5.3181E-08(3.52)	3.4903E-09(3.93)	2.2100E-10(3.98)
x= 3/4	2.0692E-06	1.4053E-07(3.88)	8.9106E-09(3.98)	5.5939E-10(3.99)
x = 1	2.3324E-06	1.5550E-07(3.91)	9.8272E-09(3.98)	6.1643E-10(3.99)

Table 5: Errors of ICFEM (k=4) at nodal points for N = 4, 8, 16, 32

N	4	8	16	32
x = 1/4	1.4911E-07	1.6962E-09(6.46)	2.4109E-11(6.14)	5.7843E-14(8.70)
x = 1/2	2.2007E-07	2.9453E-09(6.22)	4.3920E-11(6.07)	1.6509E-13(8.06)
x= 3/4	1.172E-07	2.3294E-09(6.20)	3.4766E-11(6.07)	9.2593E-14(8.55)
x = 1	1.4803E-07	1.9886E-09(6.22)	2.9543E-11(6.07)	2.1483E-13(7.10)

4. Conclusions

We consider nonlinear two-point boundary value problem using the Interpolated Coefficients Finite Element Method (ICFEM). The errors for ICFEM k = 1, 2, 3, 4, 5) of N = 4, 8, 16, 32 at the node are listed in Tables 2-5 and Figure 2, respectively. The error convergence order of the 3rd ICFEM at the nodes is basically the normal finite element convergence order k + 1, and k = 2, 4, the error convergence order of the 4th ICFEM at the node is generally higher than k + 1, almost reaching the 2k order super. The higher dimension situation will be the furture work.

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