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The SF Rule of Classical Probability: An Exploration of the "Gambler's Paradox"

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Abstract: This paper uses the "gambler's paradox", an event of classical probability origin, as an entry point, and use graphical and observational methods to generalize the SF rule. The SF rule is used to explain the compound events, to demonstrate the formal guarantee of the SF rule, and to deduce the detailed steps of using the SF rule. As an important rule for selecting basic events in classical probability, the SF rule should be described in the probability foundation.

1. Introduction

Probability theory has its origins in the 1654 correspondence between Fermat and Pascal, dealing with the familiar problem of gambling capital. [1] This event gave rise to classical probability, which is also the first basic model of probability cognition.

The problem of gambling capital has been repeatedly questioned and discussed by some well-known mathematicians, such as Pascal, Chevalier, Leibniz, etc. Although Fermat solved this problem by relying on classical probability model, Pascal cited Roberval's view to question that if the gambling game continues, it may not conform to Fermat's probability formula. In other words, Fermat's probability formula requires people to accept 'fictitious' probability possibility, which will not happen in real gambling.

This problem involves the sample space problem of probability occurrence. Until 2017, some scholars in the United States even considered this gambler's paradox not as a problem solved at the mathematical or technical level, but as a metaphysical problem that involves causality and the generality of probabilistic reasoning. [2] In fact, from the perspective of classical probability, this problem can be completely solved by the formal rule of sample space (called SF rule in this paper), without resorting to metaphysical analysis like previous scholars. SF rule is so basic and important, but it is ignored by a large number of mathematical statistics textbooks, which leads to fuzziness in the choice of elementary events. Probability and statistics professionals use SF rule imperceptibly when solving classical probability problems, but few people describe it in detail.

The framework of this paper is as follows: In the first part, this paper introduces the problem of gambling cost and analyzes it with SF rule to answer Roberval's query on Fermat's classical probability model. In part 2, complex composite events and basic events with unequal probability are derived, and their effectiveness is verified by using SF rule. In Section 3, the formal guarantees of SF rules are argued using the idea of natural deduction, and the steps of using SF rules are summarized and illustrated with complex event cases; in Section 4, the key points of SF rule are summarized and a call is made to enhance the introduction of SF rule in textbooks of probability and mathematical statistics.

2. The Gambler's Paradox

Pascal's communication with Fermat mainly revolves around a difficult problem in the theory of opportunity, which is the well-known point problem. Suppose there are two gamblers, A and B, who are gambling in multiple rounds, and each has a fair chance to win, which is similar to a coin toss game. Before the gambling starts, the two people have equal gambling costs. The winner who wins the n rounds of the game first will take away all the gambling costs of the two people. The number of rounds A wins is a and B wins b rounds. When two people fail to complete the specified number of rounds, how should they allocate all the gambling costs?

The following is a simplified simulation of the gambler's point problem scenario. Suppose that A and B each carry 6 gold coins, a total of 12 gold coins. In the following round robin, the winner can take away all the gambling costs. Since even rounds are prone to draw, the odd round system can only be considered. Then there are two forms: (2n-1) and (2n+1). If substituted according to the natural number sequence counting rule (here 1 is used as the starting n), the former starts at 1 and the latter starts at 3. In real life, we prefer the latter, because a showdown is somewhat unfair to the loser. But in real life, such games do exist, such as the "one exam to determine the winner" in the examination system, or the World Cup finals. Therefore, it is appropriate to choose the form of (2n-1), and Fermat's formula is also selected from (2n-1). [3] Combined with the times a and b described above, the overall number of games that need to be played continuously is (2n-1) - (a+b). Calculation of Fermat gambler paradox is based on the following conditions:

(1) The total number of games to be played: M=(2n-1);

(2) A has won a rounds, B has won b rounds, then the continuous game will be played again: m = (2n-1) - (a + b);

(3) The minimum number of winning games to be played is mm, then the difference between the maximum and minimum number of games to be decided is

d= M - mm;

(4) The sample space generated until the winner is decided is 2^m

All the games to be played are generally tiebreakers, two out of three, three out of five .... The 2n-1 format is met. Assuming that A and B have not yet played any match, this a=0, b=0 situation is discussed as the extreme minimal case, and of course the level, skill and various circumstances of both are assumed to be equivalent, resulting in the following table:

Table 1: Gambler’s problem of points

a	b	n	M	mm	d	m	2^m
0	0	1	1	1	0	1	2
0	0	2	3	2	1	3	8
0	0	. . . .	. . . .	. . . .	. . . .	. . . .	. . . . . .

When a game is played to determine a winner, there is no debate, either A wins or B wins. The problem lies in the second line, when two choices are needed, the argument arises when at least 2 games and at most 3 games can be decided in a two-out-of-three system. Let's look at the following situation where A wins:

Table 2: A winning

sample space	AAA	AAB	ABB	ABA	BAA	BAB	BBA	BBB
A winning	√	√		√	√			

The ticked option is the probability of A winning, i.e. {AAA, AAB, ABA, BAA}. But Pascal cites Roberval's view that in the case of two out of three games, AAA and AAB seem redundant, because as soon as A wins 2 games, the game should be terminated immediately in a realistic gambling game, and there will be no third game, which means that AAA and AAB can be considered as AA uniformly, so that the sample space should be {AA, ABA, BAA}. Similarly, if it is B that wins, then the situation is similar, and there will not be 4 cases, but will be the case of {BB, BAB, ABB} as a substitute, so that the overall sample space should be 2^m-2=6, instead of Fermat's assumption of 2^m = 8.

Leaving aside the case of year-over-year scaling of the numerator and denominator, we can go much deeper into the issue of sample space selection alone. If A and B are not in the 0 starting position, we can see that this arbitrary deletion of options can cause even more trouble when the game continues for several rounds. It is well known that once the basic event options are reduced, then the probability ratio of the fixed sample space is bound to change. Here is a discussion of the situation with a=2 and b=1. The gamble proceeds to this point, at which point the police come and the gamble must be suspended, so how should the overall 12 coins be allocated? See the following chart for details:

Table 3: Table of five sets of three wins

a	b	n	M	㎜	d	m	2^m
2	1	3	5	3	2	2	4

The sample space at this point 2^m= {AA, AB, BA, BB}. In these 4 cases, B only accounts for 1/4, and should take 3 of the 12 coins in total, but since A has already won 2 games, and the game will end if one more game is added, then {AA, AB} can only happen in one possible case, and the overall sample space will become {A, BA, BB}. Then the probability of B winning becomes 1/3 of the total, so he should take 4 coins. At this point the paradox arises, neither A nor B will accept this vague standard allocation scheme. From the above list, it can be found that the crux of the paradox lies in the value of d. When d > 1, at least 2 (including 2) or more possibilities of winning are generated, because different judgments are made based on different criteria.

In the history of mathematics, such problems are usually ignored. Both Pascal and Leibniz believe that Fermat has successfully solved this problem. [4] Fermat's idea is that the choice of basic events here must be equal chance, when there are four cases, 2^m = {AA, AB, BA, BB}, they all have equal probability. Because every time a game is played, there are bound to be two cases, A or B. This is the same as the heads and tails of a coin, or the six sides of a die. But real life experience, again, interrupts our judgment alive. Until 2006, there were still scholars who believed that Fermat was forcing people to acknowledge illusory probabilities and admit what could not happen in reality. [5] In fact, the correspondence between Fermat and Pascal does explain this clearly, and we need to understand the deeper meaning of Fermat's classical probability formula, which means that we can use the classical probability only when all the equal possible outcomes are taken, and not to interrupt this science of formal rules with arbitrary life experiences. We say that classical probability is an abstraction of life to guide life, not that we can just change the sample space according to the change of the number of basic events. Take this topic as an example, the ratio of probability of AA: AB:BA:BB is 1:1:1:1, but if it becomes A, BA, BB, then it is 2:1:1. If we force this ratio of 2 as the possibility of only one occurrence in the result, it will become 1:1:1, and then the paradox happens. Because A and BA, BB do not occur in the same ratio, they cannot constitute the basic event, and therefore cannot constitute the sample space. To go deeper, this is actually a violation of the SF rule, which is discussed in detail in Part 3 of this paper.

While possible assumptions such as the fundamental event can resolve the gambler's paradox, do all fundamental events have the same probability of occurring, and does the SF rule still work for non-equivalent fundamental events? We move on to Part 2, which explores the more complex compound events.

3. Non-Equivalent Events

The equiprobability of Fermat's basic events guarantees the regularity of the sample space. But for beginners, this is an illusion because beginners are then exemplified with events like coin tosses and dice tosses. If one has to ask why the probability of a coin toss is 1/2 and a dice toss is 1/6, one can only say that the probability was replaced by frequency through a large number of random trials. Figure 1 below shows the distribution of 100 one-dice tosses, which shows that the probability of 1-6 points occurring is approximately in accordance with the principle of equal probability. However, the above gambler's paradox {A, BA, BB} cannot be treated as a fundamental event, or at least not at the same level, to be more precise.

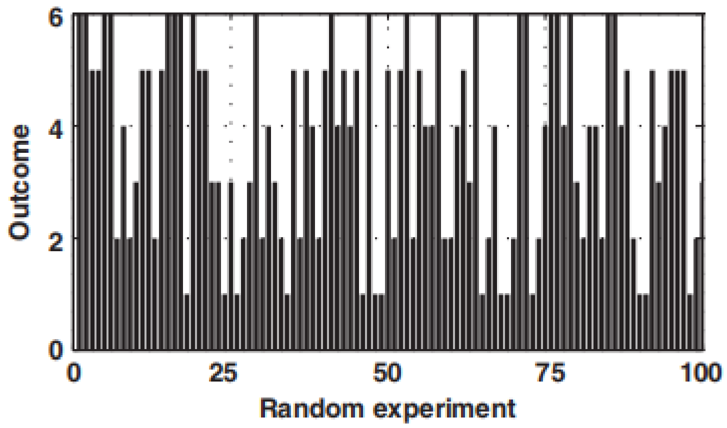

Figure 1: Distribution of 100 throws of 1 die randomized trials [6]

In other words, when selecting the basic events, we have to follow the rules of probability space 2^m to select them, and any event that violates this SF rule cannot constitute a basic event. Then the question arises, are all events equally likely? The answer is naturally not. If we go deeper, are all the basic events in the fixed probability space equally likely? This would require a detailed analysis, because it would involve the question of whether the SF rule is universal. Taking the example of throwing two dice, the probability space of throwing two dice at the same time is $N_1 \times N_2$, which not only refers to the multiplication of two numbers, but also means the number of combinations of all possible faces of two dice. Therefore, according to the SF rule, the number of basic events is also the number of combinations of two dice $n_1 \times n_2$, and it is reasonable to reason that the probability of the basic event of one die is 1/6, and the probability of two dice is 1/36. However, the reflection of realistic experience does not line up according to a simple case like 1/36, and the following is the distribution column of the two dice throws:

Table 4: List of the number of combinations of two dice

Sum of points	2	3	4	5	6	7	8	9	10	11	12
Frequency distribution	1	2	3	4	5	6	5	4	3	2	1

Does such a distribution of frequencies violate the SF rule? Actually, it does not. The question of what is a fundamental event is to be explored here. According to Pascal's definition, a fundamental event in a probability space is an indivisible event, also known as an atomic event. Since the throw of 2 dice is already a composite event, the number of combinations of 2 dice is the fundamental event, and further subdivision would defeat the purpose of the "throw of 2 dice" test. But the concept of atomic events is very important. In fact, 2 dice is just a combination of atomic events, but because the problem is limited, it is impossible to continue to divide. If we look at the occurrence of atomic events, any kind of combination of 2 dice is of course 1/36. In the case of a sum of 3 points, the frequency of one {1,2} point is 1/36, and except that there is another {2,1} point case, which is why the frequency of 2 appears in the second row and third column of Table 4 above. Therefore, the probability distribution of throwing 2 dice is as follows in Table 5:

Table 5: Columns of points and distribution of the two dice

Sum of points	2	3	4	5	6	7	8	9	10	11	12
P	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Therefore, this basic event taking problem with unequal probabilities is taken under the case of combination of atomic events with equal probability. If the atomic events are taken out in the first step according to the SF rule, and then combined in the second step according to the requirements of the problem set, the correct probability distribution result can be obtained. The following is a clearer representation in the case of the test graph:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
	(2,1)	(2,3)	(2,3)	(2,4)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)	
		(3,1)	(3,2)	(3,3)	(3,4)	(4,4)	(5,4)	(6,4)		
			(4,1)	(4,2)	(4,3)	(5,3)	(6,3)			
				(5,1)	(5,2)	(6,2)				
					(6,1)					

$N_1 \times N_2 \rightarrow 6 \times 6$

Figure 2: Two dice throwing test diagram

The graphical analysis of the above case clearly shows the importance of taking atomic events according to the form of probability space, and the rule of taking sample points according to the composition of probability space is the SF rule. As long as it is a classical probability problem, the correct conclusion can be drawn by taking sample points according to the SF rule and then making a combination that fits the question. But of course, mathematical discussions should be rigorous and normative. In the following part 3, we will prove the formal rules of the SF rule and describe the steps of its use.

4. Formal Proofs of SF Rule and Steps to Use Them

The formal proof of the SF rule relies on Laplace's definition of classical probability, [7] which specifies the equiprobability of atomic events of classical probability; it also requires the introduction of a proof method, and the natural deduction method is used here for the formal proof of the SF formal rule. The natural deduction method conforms to the general rules of deductive reasoning and facilitates the proof of SF rules by introducing any proven theorem as a precondition in the proof process. At the beginning of the proof, two preconditions, Laplace's classical probability definition and Bernoulli's law of large numbers, need to be introduced. And the proof process is as follows:

$$\mathrm{p(A)} = \frac{n}{N} \cdots\cdots\cdots\cdots\cdots\cdots \text{(Laplace classical probability definition)(1)}$$

$$\lim_{n\to\infty} p(|\frac{\mu_n}{n} - \mathrm{p}| < \varepsilon) = 1 \cdots\cdots\cdots\cdots\cdots \text{(Bernoulli's law of large numbers)(2)}$$

$$\lim_{n\to\infty} \frac{\mu_n}{n} \approx \mathrm{p} \cdots\cdots\cdots\cdots\cdots (3)$$

$$p(A) = \frac{n}{N} = \frac{n_1}{N} = \frac{n_2}{N} = \frac{n_n}{N} \cdots\cdots\cdots\cdots\cdots (4)$$

$$\sum \lim_{n\to\infty} \frac{\mu_n}{n} = \sum_1^n \frac{n_n}{N} = \sum p = \Omega \cdots\cdots\cdots\cdots\cdots (5)$$

$$\sum_1^n \frac{n_n}{N} = \frac{\sum_1^n n_n}{N} = 1 \cdots\cdots\cdots\cdots\cdots (6)$$

$$\sum_1^n n_n = \mathrm{N} \cdots\cdots\cdots\cdots\cdots (7)$$

$$n_n \sim \mathrm{N} \cdots\cdots\cdots\cdots\cdots (8)$$

(7) Similar in form

The formal rule proof here can be summarized in two points: first, the reason why the final steps (7)-(8) can be deduced for the case where n is similar to N is that step (6) gives the guarantee of a natural series, which of course is due to the case of discrete variables in the classical probability itself, and is also a derivation of the equal likelihood of atomic events; second, during the proof, N as the probability space never changes in any way, which means that using the SF rule is not possible to change N arbitrarily based on the case of atomic events, or else atomic event sampling errors will occur.

Based on the above proof steps, this paper summarizes the specific steps of using SF rule as follows.

(1) Determine whether the object of calculation is a classical probability type, and if so, determine the form of the probability space.

(2) Extract atomic events according to the form of the probability space, and the form of the composition of atomic events is consistent with the form of the probability space.

(3) Combination events are recombined based on atomic events.

(4) The form of the probability space cannot be changed in the reverse direction because the form of the combined events has changed.

The following is a more complex classical probability problem based on the SF rule steps derived by induction, i.e., the room assignment problem in classical probability. What is the probability that there are n people, each of whom is assigned to any one of the N rooms with the same probability 1/N ($n \le N$), and that there is one person in each of the specified n rooms? [8]

According to the SF rule, the probability space is determined first. Because n persons face the choice of N rooms, the probability of each person's room allocation is 1/N, n persons should have N^n possible choices of situations, and thus the probability space is N^n; the second step is to determine the atomic event, each person's choice may be $1/N^n$, which is consistent with the elements of the atomic event; the third step is to compound the atomic event according to the question, because n rooms are specified, 1 person faces n/N^n kinds of situations, and n individuals face n^n/N^n situations. From the present situation, the extracted events have exactly the same form as the probability space, i.e., the SF rule is satisfied. However, the question further requires that there is one person in each of the n rooms, which means that there cannot be two or more people in one room. Then, it is necessary to compound again according to the question set. Each person divided into rooms needs to be divided separately, i.e., different rooms for each person, when the first person chooses for n/N^n cases, and when the second person chooses again for $n-1/N^n$ cases ...... So on and so forth, then the final result will be: $n\times(n-1)\times(n-2)...\times1/N^n = n!/N^n$.

It may be questioned here that if the second person in the composite event changes to n-1 choices, should the probability space also change to N-1 choices? After all, the number of people is reduced by 1, and the number of rooms chosen is also reduced by 1. This requires using point (4) of the SF rule, which does not reverse the form of the probability space just because the form of the composite event has changed. In fact, the probability space N^n is the form of possibility after summing all atomic events. If we want to change it, we can only change the compound events according to the conditions of the question, not the probability space, otherwise it will violate the SF rule and lead to calculation errors. This is another case of life experience interfering with the mathematical form, which is similar to the gambler's paradox. Imagine how 4/5 can equal 3/4? So, this kind of thinking that subtract one from both numerator and denominator at the same time is not desirable.[9] The importance of the SF rule, which ensures the correctness of extracting the basic events in the classical probability calculation, is also verified in reverse here.

5. Conclusion

Mathematics is a formal science, an abstraction of life, reflecting the laws of nature. Mathematical ideas must not be interpreted because of changes in life experience. The gambler's paradox has already given us a hint. In fact, real life does not always correspond to the mathematical form 100%, for example, the concept of infinity is rarely used in life, and it is difficult to simulate such a situation in life, but it does not mean that the mathematical form of "∞" is wrong.

The classical probability model, the first cognitive model of probability theory, is a simple yet profound form that is the basis for in-depth study of probability theory, and has been revised by mathematicians and simplified to the form seen today. The SF rule is a guarantee of correctness for extracting basic events in the probability space, but I have collected Chinese and foreign probability theory textbooks for the past 5 years and found no detailed introduction of the SF rule in the basic section. Perhaps the textbook authors want readers to master this rule in practical exercises training. It is more likely that new mathematical concepts are used to circumvent such problems as basic event selection. For example, textbooks are now using the concepts of sample points and sample space for randomized trials, simplifying previous concepts like basic events and probability space. Since there is a random test guarantee, the so-called random is equal probability, of course all the sample points of equal probability should be taken, without regard to whether these sample points occur in real life, thus also eliminating such embarrassment as the gambler's paradox. However, in the eyes of beginners, it is necessary to explain as close as possible to the reality of life. The introduction of SF rule is equivalent to constructing a bridge of understanding between the basic events and the sample points, which can make it easier for beginners to understand the thinking pattern of classical probability in depth.

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