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A Semismooth and an Inexact Newton Method for Solving Nonsmooth Operator Equations

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Abstract: After a series of changes, many application problems in the fields of transportation, finance, economy, engineering technology and resource allocation are equivalent to solve a class of nonsmooth operator equations. In order to solve the nonsmooth operator equations and study its application in Banach space, this paper develops a semismooth Newton method and inexact-Newton method via a generalized inverse. The global, local and superlinear convergence of a semismooth Newton method are shown, the superlinear and linear convergence properties of an inexact Newton method are also proved. The present methods can be easier to perform than the previous ones in some applications, and viewed as the extensions of existing methods to solve nonsmooth operator equations.

1. Introduction

In abstract spaces, using Newton methods to deal with nonsmooth equations have become a hot topic of natural science and engineering mathematics research in recent decades. Argyros proposed the Newton method's local convergence of the second Fr´echet-derivative equations and built their solutions' weak sufficient convergence in Banach space[1]. Michael Ulbrich proved the Newton-like method's superlinear convergence for semismooth operator equations and developed a concept of semismoothness for nonsmooth superposition operators[2]. Liu and Gao proposed an inexact Newton method and developed its superlinear convergence for infinite-dimensional nonsmooth operator equations[3]. Mosi´c and Marovt presented two new outer inverses (a weighted weak core inverse and its dual one) as solutions to some equations via two Hilbert spaces' bounded linear operators[4]. Other studies on Newton methods have emerged for solving different problems of variational inequality, nonlinear complementarity and nonlinear equations[5-9]. Existing studies were mostly based on one or two factors of semismooth operator, inexact Newton method or outer inverse, we take all the factors into account when dealing with the problem of nonsmooth operator equations in Banach space. We will discuss a class of nonsmooth operator equations denoted as follows:

$$F(x)=0, \tag{1}$$

Where X and Y refer to two Banach spaces,and $F(\cdot):D\subset X\to Y$ refers to an semismooth mapping.

Many problems in applied Mathematics and engineering are solved by seeking some certain equations' solutions[10]. For instance, the state of a dynamic system is usually represented by the solution of a difference or differential equation, and its equilibrium state is decided by dealing with the equation (1).Newton method is an iteration method for solving the equation (1) well. In Banach space, when a bounded linear operator is not a bijective, its inverse operator is inexistent. While, its generalized inverse has no such defect. Therefore, we can pay attention to the study of its generalized inverse in Banach space. The existing studies have focused on a variety of the bounded linear operators' generalized inverses[11-14], for example, bounded inner inverse, bounded outer inverse and Drazin generalized inverse. In particular, it is worth mentioning that the stability of a bounded linear operator's outer inverse is well. For the practice purpose, we consider a bounded outer inverse, propose a semismooth Newton method and an inexact Newton method when dealing with the problem of nonsmooth operator equations, and prove their convergence properties of linear and superlinear based on certain assumptions.

The structure of our article is as follows: Section 2 lists some preliminary knowledge that will be often used later, Section 3 shows a semismooth Newton method to deal with equation (1) by using the bounded outer inverse and proves the corresponding convergence properties, Section 4 proposes an inexact-Newton method and establishes its superlinear and linear convergence, Section 5 states some conclusions in the ending.

2. Preliminary Knowledge

We will show some definitions and lemmas which need to be used in this article firstly. Throughout the article, $\|\cdot\|$ denotes its norm, $\pounds(X,Y)$ denotes bounded operators' Banach space from X to Y with a operator norm $\|\cdot\|$, Then it holds that

$$\forall T\in\pounds(Y,X),\quad \|T\|=\sup_{x\neq 0}\frac{\|T(x)\|}{\|x\|}=\sup_{\|x\|=1}\|T(x)\| \text{ and } 0\le\|T\|<\infty$$

Definition 1[15] Suppose that $F:D\subset X\to Y$ refers to a operator and mapping, $\partial^*F:D\mapsto(X,Y)$ refers to a set-valued mapping for an open subset D,

1) The operator F is semismooth for each point x if

$$\sup_{\Gamma\in\partial^*F(x+p)\neq 0}\|F(x+p)-F(x)-\Gamma(p)\|=o(\|p\|),\quad \text{when}\quad \|p\|\to 0,$$

And F is continuous at x.

2) The multifunction ∂^*F is called generalized differential of F.

Definition 2[11] Suppose that a bounded linear operator $T^{\#}:Y\to X$ is said to be a bounded outer inverse of T if $T^{\#}TT^{\#}=T^{\#}$; $\Omega(T)=\{T^{\#}\in\pounds(X,Y):T^{\#}\neq 0, T^{\#}TT^{\#}=T^{\#}\}$ is a set of $T^{\#}$ which is nonzero; $R(T)$ and $N(T)$ are the range and the null of T respectively.

Lemma 1[11] Let $T_0\in\pounds(X,Y)$ and $T_0^{\#}\in\pounds(Y,X)$ be a bounded outer inverse of T_0. Let $T\in\pounds(X,Y)$ hold such that $\|T_0^{\#}(T-T_0)\|<1$. Then $T^{\#}=\left(I+T_0^{\#}(T-T_0)\right)^{-1}T_0^{\#}$ is a bounded outer inverse of T. Moreover, it satisfies that $N(T_0^{\#})=N(T^{\#})$, $\|T^{\#}T_0^{\#}\|\le\frac{1}{1-\|T_0^{\#}(T-T_0)\|}$, $\|T^{\#}-T_0^{\#}\|\le\frac{\|T_0^{\#}(T-T_0)\|}{1-\|T_0^{\#}(T-T_0)\|}\|T_0^{\#}\|$ and $R(T^{\#})=R(T_0^{\#})$.

Lemma 2[11] Suppose that $T^{\#}$ is a bounded outer inverse of $T \in £(X,Y)$, then $X = R(T^{\#}) \oplus N(T^{\#}T)$ and $Y = N(T^{\#}) \oplus R(TT^{\#})$ holds.

Lemma 3[11] Suppose that $T_0^{\#}, T^{\#} \in £(Y,X)$ are both bounded outer inverses, and $T_0, T \in £(X,Y)$,then $T^{\#}(I - T_0 T_0^{\#}) = 0$ when it has $N(T_0^{\#}) \subset N(T^{\#})$.

Assumption 1 Let $\{x_k\}$ converge to a point x^* linearly and $S(x^*,\eta)$ be a neighborhood of x^*.Then $\exists \eta, p > 0, \forall \Gamma \in \partial^* F(x), \forall x \in S(x^*,\eta),\ s.t.\ N(\Gamma(x)^{\#}) = N(\Gamma(x_0)^{\#}),\ when\ \|\Gamma^{\#}\| \le p$, where $\Gamma^{\#} \in \Omega(\Gamma)$ is a bounded outer inverse .

3. A semismooth Newton Method

We develop a semismooth Newton method and its convergence properties for solving equation (1) in this section. Suppose that $\exists x_0 \in D \subset X$,$\Gamma(x_0)^{\#} \in £(Y,X)$ is a bounded outer inverse, it satisfies that $\|\Gamma(x_0)^{\#}(\Gamma(x) - \Gamma(x_0))\| < 1$ for $\forall \Gamma \in £(X,Y)$. A Newton method for dealing with the equation (1) is proposed as following:

$$x_{k+1} = x_k - \Gamma(x_k)^{\#} F(x_k), \quad \Gamma(x_k)^{\#} \in \partial^* F(x_k). \tag{2}$$

3.1. Local and Superlinear Convergence

Let the Assumption 1 hold and $F: X \to Y, D \subset X$ be semismooth, $x^* \in D$ be a solution of the equations (1). Then the iteration method given in (4) is well-defined and generated a sequence $\{x_k\}$ which converges superlinearly to a point x^* in its neighborhood.

Proof. Let $x \in S(x^*,\eta)$ with $\eta \in [0,\varepsilon]$ sufficiently small. Then, the step x_k is well defined by Assumption 1. Furthermore, according to Assumption 1, $F(x^*) = 0$ and the nonsmoothness of F ,we can obtain that

$$\begin{aligned} \|x_k - x^*\| &\le \|\Gamma(x_k)^{\#}\| \|F(x_k) - F(x^*) - \Gamma(x_k)(x - x^*)\| \\ &\le p\|F(x_k) - F(x^*) - \Gamma(x_k)(x_k - x^*)\| = o(\|x_k - x^*\|), \quad \text{when } k \to \infty. \end{aligned}$$

Its convergence is proved.

3.2. Global and Superlinear Convergence

Suppose that a semismooth mapping $F: X \to Y, D \subset X$, a bounded outer inverse $\Gamma(x_0)^{\#} \in £(Y,X)$ for $\exists x_0 \in D \subset X$, $\exists \alpha \in (0,1), \eta > 0$, then a bounded outer inverse $\Gamma(x)^{\#} \in \Omega(\Gamma(x))$ satisfies $N(\Gamma(x_0)^{\#}) = N(\Gamma(x)^{\#})$

For $\forall \Gamma(x)$. Let the condition $\|\Gamma(x_0)^{\#} F(x_0)\| \le \eta$ and

$$\|\Gamma(y)^{\#}[F(y) - \Gamma(x)(y - x) - F(x)]\| \le \alpha \|y - x\|, \quad when\ \ y = x - \Gamma(x)^{\#} F(x)$$

Hold and $S = S(x_0, r) \subseteq D$ with $r = \eta/_{1-\alpha}$. Then $\{x_k\}$ is a sequence determined by (2) , it satisfies the equality $N(\Gamma(x_k)^{\#}) = N(\Gamma(x_0)^{\#})$ and also converges to a point x^* which is a solution of $F(x)\Gamma(x_0)^{\#} = 0$ for S . In addition, the convergence rate is superlinear when the Assumption 1 holds.

Proof. First, for the sequence $\{x_k\}$ defined by formula (2), Let $k=1$, we have

$$\|x_1-x_0\|=\left\|-\Gamma(x_0)^{\#}F(x_0)\right\|\le\eta=(1-\alpha),$$

and thus $x_1\in S$. Suppose that $x_p\in S\ (p=1,2,\cdots,k)$, a bounded outer inverse $\Gamma(x_k)^{\#}$ satisfies that $N\left(\Gamma(x_k)^{\#}\right)=N\left(\Gamma(x_{k-1})^{\#}\right)=N\left(\Gamma(x_0)^{\#}\right)$, then there has the equation $\Gamma(x_k)^{\#}\left[I-\Gamma(x_{k-1})^{\#}\Gamma(x_{k-1})\right]=0$ holds by Lemma 3, and

$$\|x_{k+1}-x_k\|=\left\|-\Gamma(x_k)^{\#}F(x_k)\right\|=\left\|\Gamma(x_k)^{\#}\left[F(x_k)-\Gamma(x_{k-1})(x_k-x_{k-1})\right]-\Gamma(x_{k-1})\Gamma(x_{k-1})^{\#}F(x_{k-1})\right\|$$
$$\le\left\|\Gamma(x_k)^{\#}\left[F(x_k)-F(x_{k-1})-\Gamma(x_{k-1})(x_k-x_{k-1})\right]\right\|\le\alpha\|x_k-x_{k-1}\|\le\eta\alpha^k=r(1-\alpha)\alpha^k$$

Hence $\|x_{k+1}-x_0\|\le\sum_{i=0}^{k}\|x_{i+1}-x_i\|\le\sum_{i=0}^{k}r(1-\alpha)\alpha^i\le r$,This proved that $\{x_k\}\subseteq S$.

For $\forall k,n\in N^+$, it has $\|x_{k+1}-x_0\|\le\sum_{i=k}^{k+n}\|x_{i+1}-x_i\|\le\sum_{i=k}^{k+n}r\alpha^i(1-\alpha)\le r\alpha^k$ and $\{x_k\}\to x^*\in S$. By Lemma 1,

$$\Gamma(x_{k-1})^{\#}=\Gamma(x_0)^{\#}\left[I+\left(\Gamma(x_{k-1})-\Gamma(x_0)\right)\Gamma(x_0)^{\#}\right]^{-1}$$

Holds. Then we have

$$\left[I+\left(\Gamma(x_{k-1})-\Gamma(x_0)\right)\Gamma(x_0)^{\#}\right](x_k-x_{k-1})=-\Gamma(x_{k-1})^{\#}F(x_{k-1})\left[I+\left(\Gamma(x_{k-1})-\Gamma(x_0)\right)\Gamma(x_0)^{\#}\right]=-\Gamma(x_0)^{\#}F(x_{k-1})$$

Hence $F(x^*)\Gamma(x_0)^{\#}=0,\ as\ k\to\infty$.

By means of $N\left(\Gamma(x_k)^{\#}\right)=N\left(\Gamma(x_0)^{\#}\right)$, it is obtained that $\Gamma(x_k)^{\#}F(x^*)=0$ and $\exists p>0,\ s.t.\ \left\|\Gamma(x_k)^{\#}\right\|\le p$. Then

$$\|x_{k+1}-x^*\|=\left\|x_k-\Gamma(x_k)^{\#}F(x_k)-x^*\right\|\le\left\|F(x_k)-\Gamma(x_k)(x_k-x^*)-F(x^*)\right\|\left\|\Gamma(x_k)^{\#}\right\|=o\left(\|x_k-x^*\|\right),\quad as\quad k\to\infty.$$

Hence $\{x_k\}$ to x^* in S ,This is global and superlinear convergence proved.

4. An inexact-Newton Method

We build an inexact Newton method and its convergence properties to deal with the equation (1) in this section. The method is presented as follows:

$$x_{k+1}=x_k-T_k^{\#}F(x_k)\tag{3}$$

Where $T_k^{\#}\in\pounds(Y,X)$ is a bounded outer inverse of $T_k\in\pounds(X,Y)$

4.1. Linear Convergence

Let the Assumption 1 hold and $F:X\to Y, D\subset X$ be semismooth, $x^*\in D$ be a root of equation $\Gamma F(x)=0$, $\{T_k\}$ be a sequence and operator in $\pounds(X,Y)$ and $T_k^{\#}\in\pounds(Y,X)$ be bounded outer inverse of T_k. Also assume that $\exists\Gamma_k\in\partial^*F(x),\Gamma_k=\Gamma(x_k)\ s.t.\ N\left(\Gamma_k^{\#}\right)\in N\left(\Gamma(x_0)^{\#}\right),\Gamma_k^{\#}=\Gamma(x_k)^{\#}$ where $\Gamma_k^{\#}$ is one of its bounded outer inverses, and $\exists\varepsilon>0,\Delta>0,C_s>0,\quad s.t.\ x_0\in S(x^*,\varepsilon)$, it has that

$$\left\|T_k-\Gamma_k\right\|\le\frac{1}{3C_s\Delta}. \tag{4}$$

Then, the iterative method given in equation (3) is defined and $\{x_k\}$ produced by (3) converges linearly to point x^*.

Proof. For $\Gamma F(x^*)=0$ and $N(\Gamma_k^\#)=N(\Gamma)$, it has that

$$\Gamma_k^\# F(x^*)=0, \tag{5}$$

And $\exists p>0$, $\left\|\Gamma_k^\#\right\|\le p$. Furthermore, we have $\left\|\Gamma_k^\#\right\|\le\frac{10}{9}p$. Choose a scalar $\Delta>0$ satisfying

$$\frac{10}{9}p(1+2C_s)\le 3C_s\Delta. \tag{6}$$

We suppose that $x_k\in S(x^*,\eta)$ with $\eta\in[0,\varepsilon]$ sufficiently small. As F is semismooth, for all $\Gamma_k\in\partial^*F(x_k)$, it has

$$\left\|F(x_k)-F(x^*)-\Gamma_k(x_k-x^*)\right\|\le\frac{1}{6C_s\Delta}\left\|x_k-x^*\right\|. \tag{7}$$

Let $T_k^\#$ be the bounded outer inverse of T_k , by means of Lemma 1 and deduce, it is obtained that

$$\left\|T_k^\#\right\|\le\frac{\left\|T_k^\#\right\|}{1-\left\|T_k^\#(\Gamma_k-T_k)\right\|}\le\frac{3}{2}\Delta. \tag{8}$$

By the method (3) and the formula (5), it follows that

$$\Gamma_k^\# F(x^*)=\Gamma_k^\#\left[F(x^*)-F(x_k)+T_k(x_k-x_{k+1})\right]=0. \tag{9}$$

Then, equation (9) holds if and only if $F(x^*)-F(x_k)+T_k(x_k-x_{k+1})=0$. Therefore $x_{k+1}=x_k-\left[F(x_k)-F(x^*)\right]T_k^\#$. Thus, we have

$$\begin{aligned}\left\|x_{k+1}-x_k\right\|&=\left\|x_k-\left[F(x_k)-F(x^*)\right]T_k^\#-x^*\right\|=\left\|F(x_k)-T_k(x_k-x^*)-F(x^*)\right\|\left\|T_k^\#\right\|+\left\|\Gamma_k-T_k\right\|\left\|x_k-x^*\right\|\\&\le\frac{3}{2}\Delta\left(\frac{\left\|x_k-x^*\right\|}{6C_s\Delta}+\frac{\left\|x_k-x^*\right\|}{3C_s\Delta}\right)\le\frac{3\left\|x_k-x^*\right\|}{4C_s\Delta}\end{aligned}. \tag{10}$$

Substituting (4), (6), (7), (8) into (10) leads to

$$\left\|x_{k+1}-x_k\right\|\le\frac{3p\left[\Delta\left(\left\|x_k-x^*\right\|+\left\|x_k-x^*\right\|\right)\right]}{2}\le\frac{\left\|x_k-x^*\right\|}{2}. \tag{11}$$

By mathematical induction, the relation $x_0\in S(x^*,\varepsilon)$ and the formula (11) imply that $x_k\in S(x^*,\varepsilon)$ holds for all k. Therefore, $\left\|x_{k+1}-x^*\right\|\le\left\|x_k-x^*\right\|$ is valid for all k under the assumption $x_0\in S(x^*,\varepsilon)$, and the sequence $\{x_k\}$ converges linearly to a solution x^* of $\Gamma F(x)=0$. Thus, the linear convergence is proved.

4.2. Superlinear Convergence

Based on the Assumption 1, let $F:D\subset X\to Y$ be semismooth, $\{T_k\}$ be a operator sequence in $£(X,Y)$, $T_k^\#\in £(Y,X)$ be a bounded outer inverse of T_k, $\{x_k\}$ be the sequence produced by equation (3) with $x_k \neq x^*$ and $\{x_k\}$ converge to x^* which is a root of equation $\Gamma F(x)=0$. Then, if $\exists \Gamma_k \in \partial^* F(x), \Gamma_k = \Gamma(x_k)$ *s.t.* $N(\Gamma_k^{\#}) \in N\left(\Gamma(x_0)^{\#}\right), \Gamma_k^{\#} = \Gamma(x_k)^{\#}$ where $\Gamma_k^{\#}$ is one of its bounded outer inverses, we have that

$$\lim_{k\to\infty} \frac{\|(x_{k+1}-x_k)(\Gamma_k - T_k)\|}{\|x_{k+1}-x_k\|} = 0 \tag{12}$$

Holds and $\{x_k\}$ converges superlinearly to the solution x^*.

Proof. For convergence point x^*, we can let $x \in S(x^*, \eta)$ with $\eta \in [0, \varepsilon]$ sufficiently small. Thus, put $e_k = x_k - x^*$, one has that the sequence $\{e_k\}$ converges to zero. Since $N(\Gamma_K^{\#}) = N(\Gamma)$ and $\Gamma F(x)=0$, one has that

$$\Gamma_k^{\#} F(x^*) = 0\,, \tag{13}$$

And $\exists p > 0$ such that

$$\|\Gamma_k^{\#}\| \le p. \tag{14}$$

By means of the inexact Newton method given by equation (3), it will obtain

$$\Gamma_k^{\#} F(x^*) = \Gamma_k^{\#}\left[T_k(x_k - x_{k+1}) + F(x^*) - F(x_k)\right] = \Gamma_k^{\#}\left\{(T_k - \Gamma_k)(x_k - x_{k+1}) - \left[F(x_k) - F(x^*) - \Gamma_k e_k\right] - \Gamma_k e_{k+1}\right\}$$

From (13), it implies that

$$\Gamma_k^{\#}\left\{(T_k - \Gamma_k)(x_k - x_{k+1}) + \left[\Gamma_k e_k + F(x^*) - F(x_k)\right] - \Gamma_k e_{k+1}\right\} = 0\,. \tag{15}$$

Then the equation (15) holds if and only if

$$(T_k - \Gamma_k)(x_k - x_{k+1}) + \left[\Gamma_k e_k + F(x^*) - F(x_k)\right] - \Gamma_k e_{k+1} = 0\,. \tag{16}$$

For any $x_k \in S(x^*, \varepsilon)$, F is semismooth, thus

$$\left\|F(x_k) - F(x^*) - \Gamma_k e_k\right\| = o(\|e_k\|), \quad as \quad k \to \infty\,. \tag{17}$$

By formulas (12), (14), (16) and (17), it follows that

$$\|e_k\| \le o(\|x_{k+1} - x_k\|) + o(\|e_k\|) \le o(\|e_k\| + \|e_{k+1}\|)\,.$$

This means that $\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|} = 0$. i.e. $\{x_k\}$ converges superlinearly to the point x^*. This completes the proof.

5. Conclusions

In this study, we theoretically developed an inexact and a semismooth Newton method with their linear and superlinear convergence properties for dealing with a class of nonsmooth operator equations (1) based on some conditions in Banach space. The proposed methods are a useful supplement to the development of abstract theory. They can become a very powerful and effective tool to solve nonsmooth operator equations in practical applications.

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