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A Stochastic Collocation Methods to 1D Maxwell's Equations with Uncertainty

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Abstract: In this paper, a stochastic collocation method is considered for one-dimensional Maxwell equations with uncertainty. The random inputs of model problem comes from the dielectric constant, magnetic permeability, and the initial and boundary conditions. We first prove the regularity of the solution of one-dimensional Maxwell equations. Then the convergence of our numerical approach is verified. Further some relevant numerical examples are implemented to support the analysis.

1. Introduction

In the study of complex physical or engineering problems, there are always some uncertain factors related to physical or engineering problems, such as model parameters, boundary and initial data, random interference, regional irregularities, etc. In this case, for obtaining reliable numerical predictions, one has to include uncertainty quantification due to the random input data. Recently, there have some numerical methods for solving of partial differential equations with random inputs, such as Monte Carlo and sampling based methods [1-3], perturbation methods [1], the generalized polynomial chaos (gPC) methods [1], etc.

Now, the gPC method has been widely used to analyze PDEs with random inputs. Gottlieb and Xiu made the first attempt by considering a simple model of a scalar wave equation with random wave speeds [4]. Tang and Zhou proposed some rigorous regularity analysis for the same problem and demonstrated the convergence of the stochastic collocation methods [5].

In this work, we consider the maxwell equation with random inputs using the s-tochastic collocation methods. Collocation methods have been studied and used in different disciplines for uncertainty quantification (see, e.g., [6-9]). Following the methods introduced by [6], we use the roots of the next higher order polynomial as the points. A Lagrange interpolation of the solution $w(x, y)$ can be written as

$$I^N w(x,y)=\sum_{k=1}^{N}\tilde{w}_k(x)F_k(y), \qquad (1)$$

Where

$$F_k \in P_N, F_i\ (y_k)=\delta_{ik}, 1\le i,k\le N, \qquad (2)$$

are the Lagrange interpolation polynomials, and $\tilde{\omega}_k(x)=\omega(x,y_k), 1\le k\le n$ is the value of w at the given node $\{y_k\}\in\Theta$. In this work, we will apply the Lagrange interpolation approach to the maxwell equation with random inputs:

$$\partial tH(x, t; y(\omega)) = \mu(y(\omega))\partial xE(x, t; y(\omega)), x\in D \equiv (-1, 1), t > 0,$$

$$\mu(y(\omega)) > 0, \partial tE(x, t; y(\omega)) = \varepsilon(y(\omega))\partial xH(x, t; y(\omega)), x\in D \equiv (-1, 1), t > 0, \varepsilon(y(\omega)) > 0, \quad (3)$$

$$E(x, 0; y(\omega)) = E0(x; y(\omega)), H(x, 0; y(\omega)) = H0(x; y(\omega))\ x \in D \quad (4)$$

Let (Ω, A, P) be a complete probability space. Here **Ω** is the set of outcomes, A is the σ-algebra of events, P is a probability measure, and y is a random variable. Let ρ(y) : Γ → R+ be the probability density functions of the random variable y(ω), ω ∈ **Ω**, and its image Γ=y(**Ω**) R be intervals in R. In what follows, for simplicity, we just omit the symbol ω and assume that y is in the parametric space Γ=[-1, 1]. A well-posed set of boundary conditions is given by:

$$E(-1, t; y) = EL(t; y);\ H(-1, t; y) = HL(t; y)$$

$$E(1, t; y) = ER(t; y);\ H(1, t; y) = HR(t; y) \quad (5)$$

Eqs.3-5 complete the set up of the problem. We now solve problem (3-5) by using the Lagrange interpolation approach. We first choose a set of Gauss-collocation-points $\{y_i\}_{i=1}^N$, that is, $\{y_i\}_{i=0}^N$ are the roots of some polynomial Φ_{N+1}. We then solve the following system of equations:

$$\partial tH(x, t; y_j) = \mu(y_j)\partial xE(x, t; y_j)$$

$$\partial tE(x, t; y_j) = \varepsilon(y_j)\partial xH(x, t; y_j) \quad (6)$$

Note that with the collocation method the boundary conditions and the initial conditions can be proposed easily, which is not the case in the Galerkin methods [7]. More precisely, we have together with the initial condition.

$$E(-1, t; y_j) = EL(t; y_j);\ H(-1, t; y_j) = HL(t; y_j)$$

$$E(1, t; y\, y_j) = ER(t; y_j); \qquad H(1, t; y_j) = HR(t; y_j) \quad (7)$$

$$E(x, 0; y_j) = E0(x; y_j);\ H(x, 0; y_j) = H0(x; y_j) \quad (8)$$

The approximation solution for the original problem(3-5) is given by

$$E^N(x,t;y) = I_N^y E = \sum_{k=0}^{N} E(x,t;y_k)F_k(y),$$

$$H^N(x,t;y) = I_N^y H = \sum_{k=0}^{N} H(x,t;y_k)F_k(y), \quad (9)$$

Where F_k^y are the standard Lagrange interpolation polynomials defined by (2).

2. Regularity in Various Spaces

2.1 Regularity in H^1

Following [1], if u∈$L^2 \otimes H^k(D)$, then u(., y, t) ∈$H^k(D)$ a.e. on Γ and u(x, t, .) ∈ $L^2(\Gamma)$ a.e. on

D. Moreover, we have (for every fixed t < T) the isomorphism

$$L^2 \otimes H^k(D) \simeq L^2(\Gamma; H^k(D)) \simeq H^k(D; L^2(\Gamma))$$

With the definitions

$$L^2(\Gamma, H^k(D)) = \left\{ \Gamma \times D \to R \middle| v \textit{ is strongly measurable and } \int_\Gamma \|v(.,y,t)\|^2_{H^k(D)} dx < +\infty \right\},$$

And

$$\mathrm{H}^k(D, L^2(\Gamma)) = \Big\{ \Gamma \times D \to R \Big| v \textit{ is strongly measurable and } \forall |\alpha| \le k, \exists \partial_\alpha(v) \in L^2(\Gamma) \otimes L^2(D),$$
$$\int_\Gamma \int_D \partial_\alpha(v)\varphi(x,y)dxdy = (-1)^{|\alpha|} \int_\Gamma \int_D v\partial_\alpha \varphi(x,y)dxdy, \forall \varphi \in C_0^\infty(\Gamma \times D) \Big\}.$$

We also denote

$$\Gamma+ = \{y | y \in \Gamma, \text{ and } \varepsilon(y) \ge \mu(y)\}, \ \Gamma- = \{y | y \in \Gamma, \text{ and } \varepsilon(y) < \mu(y)\}.$$

With the above definitions, we now introduce the following lemma.

Lemma 2.1. Consider the problem (3-5).If the following conditions are satisfied:

$$\int_\Gamma \int_{\mathrm{D}} \rho(y)((\partial_x E_0(x;y))^2 + (\partial_x H_0(x;y))^2)dxdy < +\infty, \tag{10}$$

$$\left| \int_0^{\mathrm{T}} \int_{\Gamma^+} \frac{\rho(y)}{\varepsilon(y)} (\partial_t E_R . \partial_t H_R - \partial_t E_L . \partial_t H_L) dydt \right| < +\infty, \tag{11}$$

$$\left| \int_0^{\mathrm{T}} \int_{\Gamma^-} \frac{\rho(y)}{\mu(y)} (\partial_t E_R . \partial_t H_R - \partial_t E_L . \partial_t H_L) dydt \right| < +\infty,$$

Then

$$\int_\Gamma \int_{\mathrm{D}} \rho(y)(E_x{}^2 + H_x{}^2)dxdy < C(T), \quad 0 < t \le T, \tag{12}$$

Where $\rho(y) > 0$ is the probability distribution function and C(T) is a positive constant depending on T.

Proof. It follows from the governing equation (3) that

$$\partial t(E_x^2) = 2Ex \cdot Etx = 2\varepsilon(y)ExHxx, \ x \in D, \ t > 0 \tag{13}$$

$$\partial t(H_x^2) = 2Hx \cdot Htx = 2\mu(y)HxExx, \ x \in D, \ t > 0 \tag{14}$$

If *HxExx* and *ExHxx* are positive, which leads to

$$\partial_t \int_{\mathrm{D}} \rho(y)(E_x{}^2 + H_x{}^2)dx = 2\int_{\mathrm{D}} \rho(y)(\varepsilon(y)E_x E_{xx} + \mu(y)H_x H_{xx})dx$$

$$\le \begin{cases} 2\int_{\mathrm{D}} \rho(y)\varepsilon(y)(E_x H_x)_x dx, & if \ \varepsilon(y) \le \mu(y), \\ 2\int_{\mathrm{D}} \rho(y)\mu(y)(E_x H_x)_x dx, & if \ \varepsilon(y) > \mu(y), \end{cases}$$

$$=\begin{cases} 2\rho(y)/\varepsilon(y)(\partial_t E_R\cdot\partial_t H_R-\partial_t E_L\cdot\partial_t H_L), & if\ \ \varepsilon(y)\le\mu(y), \\ 2\rho(y)/\mu(y)(\partial_t E_R\cdot\partial_t H_R-\partial_t E_L\cdot\partial_t H_L), & if\ \ \varepsilon(y)>\mu(y). \end{cases}$$

The above result together with (13) and (14), yields

$$\frac{d}{dt}\int_\Gamma\int_D\rho(y)(E_x{}^2+H_x{}^2)dxdy\le 2\int_{\Gamma^+}2\rho(y)/\varepsilon(y)(\partial_t E_R\cdot\partial_t H_R-\partial_t E_L\cdot\partial_t H_L)dy$$
$$+2\int_{\Gamma^-}2\rho(y)/\mu(y)(\partial_t E_R\cdot\partial_t H_R-\partial_t E_L\cdot\partial_t H_L)dy.$$

The desired estimate (12) is obtained by integrating the above inequality with respect to t and by using the assumption (10).

Theorem2.1. Consider the problem 3-5. Assume that there exists a constant C such that

$$max\,\{|\varepsilon^J(y)|,|\mu^J(y)|\}\le C, almost\ everywhere\ in\ \Gamma, \qquad (15)$$

i.e., ε(y), μ(y) is bounded in the distribution sense in Γ. If the assumption (10) holds and furthermore if

$$\int_\Gamma\int_D\rho(y)((\partial_y E_0(x;y))^2+(\partial_y H_0(x;y))^2)dxdy<+\infty,$$
$$\left|\int_0^T\int_{\Gamma^+}\rho(y)\varepsilon(y)(\partial_y E_R.\partial_y H_R-\partial_y E_L.\partial_y H_L)dydt\right|<+\infty, \qquad (16)$$
$$\left|\int_0^T\int_{\Gamma^-}\rho(y)\mu(y)(\partial_y E_R.\partial_y H_R-\partial_y E_L.\partial_y H_L)dydt\right|<+\infty,$$

Then

$$\int_\Gamma\int_D\rho(y)(E_x{}^2+H_x{}^2)dxdy<C(T),\quad 0<t\le T, \qquad (17)$$

Where C(T) is a finite number depending on T.

Proof. Differentiating both sides of (3) with respect to y gives

$$(Ey)t = \varepsilon j(y)Hx + \varepsilon(y)(Hy)x,$$

Which yields

$$(E_y^2)t = 2\varepsilon j(y)HxEy + 2\varepsilon(y)Ey(Hy)x.$$

Similarly,

$$(H_y^2)t = 2\mu j(y)HyEx + 2\mu(y)Hy(Ey)x.$$

Integrating the above equation with respect to x leads to

$$\partial_t\int_D\rho(y)(E_y{}^2+H_y{}^2)dx = 2\int_D\rho(y)(\varepsilon'(y)E_yH_x+\mu'(y)H_yE_x)dx$$
$$+2\int_D\rho(y)(\varepsilon(y)E_yH_{yx}+\mu(y)H_yE_{yx})dx$$
$$\le 2\int_D\rho(y)\varepsilon'(y)E_yH_xdx+2\int_D\rho(y)\mu'(y)H_yE_xdx$$

$$+\begin{cases} 2\rho(y)\varepsilon(y)(\partial_y E_R \cdot \partial_y H_R - \partial_y E_L \cdot \partial_y H_L), \ if \ \varepsilon(y) \le \mu(y), \\ 2\rho(y)\mu(y)(\partial_y E_R \cdot \partial_y H_R - \partial_y E_L \cdot \partial_y H_L), \ if \ \varepsilon(y) > \mu(y). \end{cases}$$

Which yields

$$\frac{d}{dt}\int_\Gamma \int_D \rho(y)(E_y^2 + H_y^2)dxdy \le C\int_\Gamma \int_D \rho(y)(E_x^2 + H_x^2 + E_y^2 + H_y^2)dxdy$$

$$+2\int_{\Gamma^+} 2\rho(y)\varepsilon(y)(\partial_y E_R \cdot \partial_y H_R - \partial_y E_L \cdot \partial_y H_L)dy$$

$$+2\int_{\Gamma^-} 2\rho(y)\mu(y)(\partial_y E_R \cdot \partial_y H_R - \partial_y E_L \cdot \partial_y H_L)dy.$$

Where the boundedness assumption of ε(y) and μ(y) are used. The desired estimate (17) follows from Lemma 2.1, Gronwall inequality and the assumption (16).

Remark 2.1. It is worthwhile to point out that, the modifification of both the boundary and the initial data will lead to a higher regularity of the solutions for the problem 1−3. Actually, under some appropriate assumptions on the boundary and initial conditions, inspired by Theorem 2.1, we can get the following regularity results for the k-th derivatives of the solutions, i.e.,

$$\int_\Gamma \int_D \rho(y)((\frac{\partial^k E}{\partial y^k})^2 + ((\frac{\partial^k H}{\partial y^k}))^2)dxdy < C(T), \quad 0 < t \le T.$$

3. Convergence of the Collocation Method

Given a function f, its expectation is defined by

$$\tilde{E}[f] = \int_\Gamma \int_D \rho(y) f(x,y)dxdy$$

And its mean square is defined by

$$M[f] = \left(\int_\Gamma \int_D \rho(y) f(x,y)^2 dxdy\right)^{1/2}.$$

Lemma 3.1. ([2], p.289. Estimates for the interpolation error.) Assume a given function $\omega(y)$ satisfies $\omega^{(m)}$ L^2 (1, 1) and denote I_N ω its interpolation polynomial associated with the $(N + 1)$-point Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{y_i\}_{i=1}^N$, namely,

$$I_N\omega(y) = \sum_{i=0}^N \omega(y_i)F_i(y). \tag{18}$$

Then for m ≤ N the following estimate holds

$$\|\omega - I_N\omega\| \le CN^{-m}\|\omega\| \tag{19}$$

Theorem 3.1. Let E, H be the solution of 3-5 and E^N, H^N be the stochastic collocation solution of 9. If the assumptions in Theorem 2.1 are satisfied, then the following estimates on the mean-square and mean errors hold:

$$\mathrm{ems}(E - E_N) := M\,[E - E_N] \le C(T)N^{-1}, \qquad 0 < t \le T, \tag{20}$$

$$\mathrm{emean}(E - E_N) := \tilde{E}\,[E - E_N] \le C(T)N^{-1}, \ 0 < t \le T, \tag{21}$$

Similarly,

$$\text{ems}(H - H_N) := M\,[H - H_N] \le C(T)N^{-1}, \quad 0 < t \le T,$$
$$\text{emean}(H - H_N) := \tilde{E}[H - H_N] \le C(T)N^{-1}, \quad 0 < t \le T,$$

Where C(T) is a constant depending on T but independent of N.
Proof. For any fixed x, it follows from Lemma 3.1 that

$$\int_\Gamma \rho(y)\left(E(x,t;y) - E^N(x,t;y)\right)^2 dy \le CN^{-2}\int_\Gamma \rho(y)E_y^2 dy. \tag{22}$$

Integrating the above inequality with respect to x and using Theorem 2.1 yield the desired estimate (20). As for (21), it follows from a standard inequality $\|\omega\|_{L^1} \le C\|\omega\|_{L^2}$.

Remark 3.1. It is noted that the regularity of the solutions for our model problem can be much better when the regularity of both the boundary and initial conditions is good enough according to Remark 2.1. As a result, under some appropriate assumptions for the boundary and initial conditions, we can obtain a similar estimate to those in Theorem 3.1 for a general k with k ≥ 1. It implies that we can acquire the exponential convergence rate when the boundary and initial data are smooth.

4. Numerical Examples

In this section we present some numerical examples to support the theoretical results derived above. In all computations, ε = 1, μ is a random variable uniformly distributed and the corresponding simple points are the Legendre-Gauss points.

4.1 Example with H^1, H^2, H^3 in the Random Space

Consider the following problem:

$$\partial tH = \mu\partial xE,\ t > 0,\ 1 < \mu < 3,$$
$$\partial tE = \varepsilon\partial xH,\ t > 0,\ \varepsilon = 1,$$

With the following three initial conditions

$$\begin{cases} E(x,0;\mu) = 4\,\text{sgn}(\mu-2)(\mu-2), & -1 < x < 1, 1 < \mu < 3, \\ H(x,0;\mu) = -\cos(\frac{\pi}{2}(x+1))4\,\text{sgn}(\mu-2)(\mu-2), & -1 < x < 1, 1 < \mu < 3; \end{cases}$$

$$\begin{cases} E(x,0;\mu) = 4\,\text{sgn}(\mu-2)(\mu-2)^2, & -1 < x < 1, 1 < \mu < 3, \\ H(x,0;\mu) = -\cos(\frac{\pi}{2}(x+1))4\,\text{sgn}(\mu-2)(\mu-2)^2, & -1 < x < 1, 1 < \mu < 3; \end{cases}$$

$$\begin{cases} E(x,0;\mu) = 4\,\text{sgn}(\mu-2)(\mu-2)^3, & -1 < x < 1, 1 < \mu < 3, \\ H(x,0;\mu) = -\cos(\frac{\pi}{2}(x+1))4\,\text{sgn}(\mu-2)(\mu-2)^3, & -1 < x < 1, 1 < \mu < 3; \end{cases}$$

The corresponding boundary conditions are

$$\begin{cases} E(-1,t;\mu) = 4\,\mathrm{sgn}(\mu-2)(\mu-2), \\ E(1,t;\mu) = 4\,\mathrm{sgn}(\mu-2)(\mu-2), \\ H(-1,t;\mu) = -\cos(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2), \\ H(1,t;\mu) = \cos(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2); \end{cases}$$

$$\begin{cases} E(-1,t;\mu) = 4\,\mathrm{sgn}(\mu-2)(\mu-2)^2, \\ E(1,t;\mu) = 4\,\mathrm{sgn}(\mu-2)(\mu-2)^2, \\ H(-1,t;\mu) = -\cos(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2)^2, \\ H(1,t;\mu) = \cos(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2)^2; \end{cases}$$

$$\begin{cases} E(-1,t;\mu) = 4\,\mathrm{sgn}(\mu-2)(\mu-2)^3, \\ E(1,t;\mu) = 4\,\mathrm{sgn}(\mu-2)(\mu-2)^3, \\ H(-1,t;\mu) = -\cos(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2)^3, \\ H(1,t;\mu) = \cos(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2)^3; \end{cases}$$

It can be checked that the exact solutions for the above three initial boundary value problems are:

$$\begin{cases} E(x,t;\mu) = \frac{1}{\sqrt{\mu}}\sin(\frac{\pi}{2}(x+1))\sin(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2), \\ H(x,t;\mu) = -\cos(\frac{\pi}{2}(x+1))\cos(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2); \end{cases}$$

$$\begin{cases} E(x,t;\mu) = \frac{1}{\sqrt{\mu}}\sin(\frac{\pi}{2}(x+1))\sin(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2)^2, \\ H(x,t;\mu) = -\cos(\frac{\pi}{2}(x+1))\cos(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2)^2; \end{cases}$$

$$\begin{cases} E(x,t;\mu) = \frac{1}{\sqrt{\mu}}\sin(\frac{\pi}{2}(x+1))\sin(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2)^3, \\ H(x,t;\mu) = -\cos(\frac{\pi}{2}(x+1))\cos(\frac{\pi\sqrt{\mu}}{2}t) + 4\,\mathrm{sgn}(\mu-2)(\mu-2)^3; \end{cases}$$

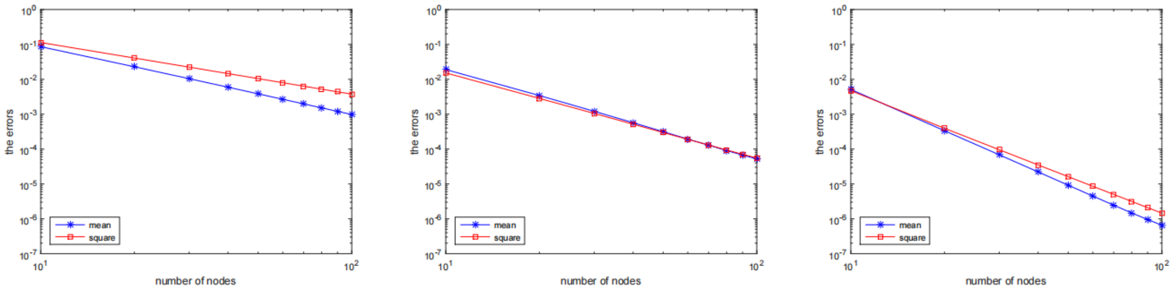

Figure 1: Example of Section 4.1: mean-square errors for E with different regularity.

Which can be verified to belong to H1, H2, H3 respectively. In fact, the initial conditions given above only belong to H1, H2, H3 respectively. Fig.1 presents the mean-square and mean errors against the number of nodes. It is clear from Fig.1 and Fig.2 that the corresponding convergence rates for the mean-square errors are 1, 2, and 3, respectively, which agrees well with the theoretical predictions. The rate for the mean errors seems better than the theoretical predictions, which implies that the estimate may not be sharp for the mean errors.

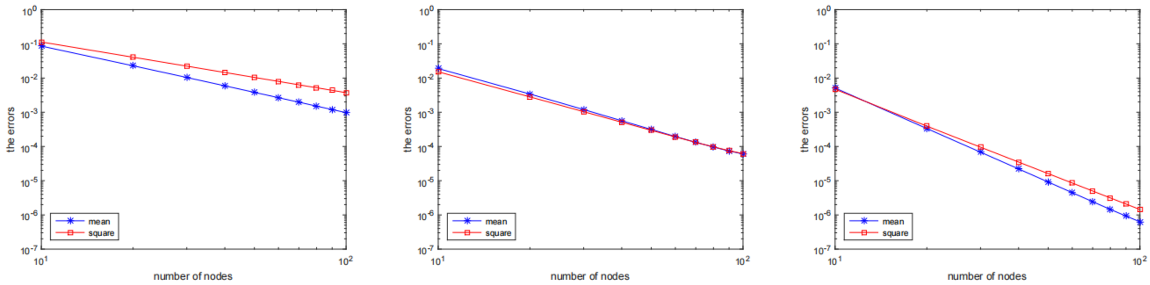

Figure 2: Example of Section 4.1: mean-square errors for H with different regularity.

4.2 A Smooth Problem

Consider the following problem:

$$\partial tH = \mu\partial xE,\ t > 0,\ 1 < \mu < 3,$$
$$\partial tE = \varepsilon\partial xH,\ t > 0,\ \varepsilon = 1,$$

With the following three initial conditions

$$\begin{cases} E(x,0;\mu) = 0, \\ H(x,0;\mu) = -\cos(\frac{\pi}{2}(x+1)); \end{cases}$$

The corresponding boundary conditions are

$$\begin{cases} E(-1,t;\mu) = 0, \\ E(1,t;\mu) = 0, \\ H(-1,t;\mu) = -\cos(\frac{\pi\sqrt{\mu}}{2}t), \\ H(1,t;\mu) = \cos(\frac{\pi\sqrt{\mu}}{2}t); \end{cases}$$

It can be checked that the exact solution for the above initial-boundary value problems are:

$$\begin{cases} E(x,t;\mu) = \frac{1}{\sqrt{\mu}}\sin(\frac{\pi}{2}(x+1))\sin(\frac{\pi\sqrt{\mu}}{2}t), \\ H(x,t;\mu) = -\cos(\frac{\pi}{2}(x+1))\cos(\frac{\pi\sqrt{\mu}}{2}t); \end{cases}$$

It is clear from Fig.3 that the exponential rate of convergence can be obtained.

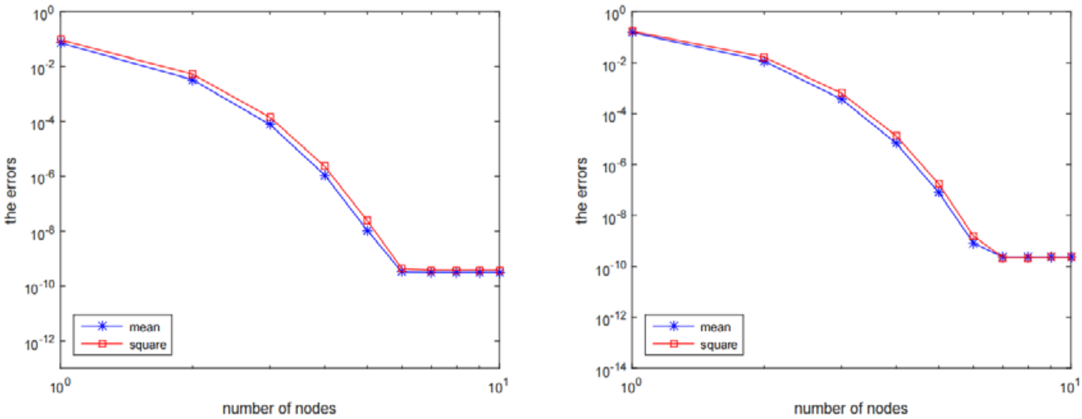

Figure 3: Example of Section 4.2: Mean-square errors for a smooth solution.

5. Conclusion

In this work, we first give some regularity results to the Maxwell equation with random inputs. The implementation of the stochastic collocation method seems convenient to handle nonlinear or more complicated problems. Stochastic methods for Maxwell equation with random inputs are still in the early stage of development. This paper provides a preliminary investigation on the stochastic collocation method for 1-dimension Maxwell equation. It has been demonstrated that the rate of convergence depends not only on the initial data and boundary conditions, but also on the random terms.The higher dimension situation will be the furture work.

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