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# *Maintenance Decision Optimization of Multi-State System Considering Imperfect Maintenance*

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***Abstract:*** In the maintenance decision-making problem of a multi-state system, it is more in line with the actual situation to consider imperfect maintenance. However, under the influence of imperfect maintenance, the failure mechanism and maintenance process of the multi-state system are more complicated. Further discussion is needed about how to deal with the relationship between the states of various components and the overall state of the system, and how to make the optimal maintenance strategy. To address this issue, this paper proposes a multi-state system maintenance decision optimization method considering imperfect maintenance. Firstly, an imperfect maintenance model is established based on quasi-renewal theory. Secondly, the mapping relationship between the state of the components and the system is decoupled combining universal generating function with the non-homogeneous continuous-time Markov model, the reliability evaluation of the multi-state system is realized. Finally, the maintenance decision optimizing model considering imperfect maintenance is constructed, the optimal maintenance strategy of the multi-state system is obtained by optimizing the number of imperfect maintenance actions to maximize the expected benefit of the whole life cycle.

## 1. Introduction

After being put into use, the aviation equipment will deteriorate due to its own reasons and the external environment. Compared with the traditional two-state system, aviation equipment is a complex system composed of multiple components, and its overall output state will gradually decrease with the failure and deterioration of components. The system may have two or more working (failure) states or performance levels, that is, multi-state characteristics. For multi-state repairable systems, a series of different maintenance actions can restore the performance level to a normal operating state. Therefore, it is a very valuable research topic for engineering practice to make an economical and effective maintenance decision to improve the reliability and economy of the multi-state system.

For the maintenance decision of a multi-state system, scholars have done a lot of research. Zhang and Wang [1] studieda class of simple deteriorating systems with k+1 states, including k failure states and one working state. The system contains only one degraded component with one maintenance group. Wang et al. [2] proposed a predictive maintenance model for multi-state systems. On this basis, a control limit maintenance strategy was proposed, under which the system was regularly checked to determine if the state of the systemexceeded the limit, and took actions of "do nothing, repair and replace". Chen et al. [3] took the multi-state system with low-priority and high-priority components as the research object, anddiscussed decentralized maintenance by introducing incomplete state observation. Attia et al. [4] gave a complete analysis process considering a four-state system. Steady-state availability was established by Markov process, and different warranty and preventive maintenance strategies werestudied. Zhang et al.[5] considered the coupling relationship between components and proposed an approximate method to analyze the state of the system, which was combined with the simulation-based method to optimize the maintenance decision of the multi-state system.

Based on the above discussion, most of the existing studies of multi-state systems considering maintenance are assumed to be analyzed at the system level or the component level,and the maintenance effect is assumed to be "as good as new" or "as bad as old" to simplify the modeling and calculation process of maintenance decision optimization. However, the maintenance effect is more inclined to imperfect maintenance, and the failure mechanism of a multi-state system is also more complicated in practice. The overall output state of the system will be affected by the states of its components together with the physical configuration between the various components. Further discussion about the relationship between the state of each component and the overall state of the system is required.

Therefore, this paper proposes a maintenance decision optimization model considering imperfect maintenance for multi-state systems.The imperfect maintenance model is establishedbased on the quasi-renewal theory, and the state probability function is used to describe the component state after maintenance. Then, the non-homogeneous continuous-time Markov model in which the state transition probability intensity changes with time is adopted to analyze the multi-state systemconsidering the relevance betweenthe multi-state components and the multi-state system. Finally, by optimizing the number of imperfect maintenance actions carried out before the replacement of the multi-state system, the expected benefit per unit of time is maximized and the maintenance decision is optimized.

## 2. Imperfect maintenance modeling based on the quasi-renewal theory

Based on the quasi-renewal theory, the changing trend normal operating time and maintenance time of the multi-state system and its components are discussed, the imperfect maintenance modeling is then analyzed. In order to be more visual and intuitive, the life cycle of the system and its components (the intersection of work and maintenance) is shown in Fig. 1.

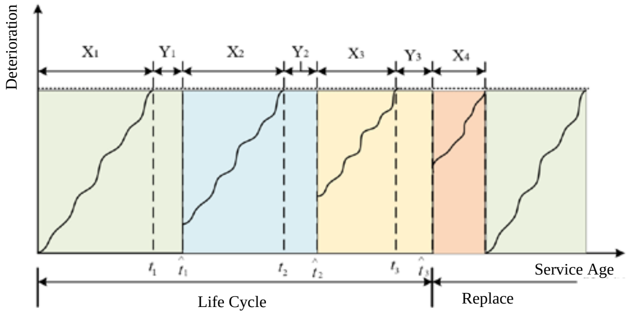


Figure 1: Schematic diagram ofthe component life cycle

The application of quasi-renewal theory in maintenance analysis meets the actual engineering requirements of a multi-state system. Because it has the advantage of describing the real characteristics of the maintenance process compared with the renewal theory, which can only describe the perfect maintenance process.

Therefore, this paper introduces the normal operating time decreasing factor $\alpha$ ($\alpha < 1$)and the maintenance time increasing factor $\beta$ ($\beta > 1$) throughout the life cycle are introduced to model the imperfect maintenance process. Based on the quasi-renewal theory, the distribution function, probability density function, and failure rate of the normal operatingtime of the multi-state component in the $i$-th maintenance cycle are expressed as follows:

$$F_{X_i}(t) = F_{X_1}(\frac{t}{\alpha^{i-1}}) \tag{1}$$

$$f_{X_i}(t) = \frac{f_{X_1}(\frac{t}{\alpha^{i-1}})}{\alpha^{i-1}} \tag{2}$$

Similar to the analysis of operating time, the first maintenance time $Y_1$and the subsequent maintenance time$Y_i$ also have similar characteristics in the probability density function and distribution function, which are expressed as follows:

$$H_{Y_i}(t) = H_{Y_1}(\frac{t}{\beta^{i-1}}) \tag{3}$$

$$h_{Y_i}(t) = \frac{h_{Y_1}(\frac{t}{\beta^{i-1}})}{\beta^{i-1}} \tag{4}$$

## 3. Reliability modeling of the multi-state system based on non-homogeneous continuous-time Markov

The degradation process of multi-state systems and components is not only related to the current state but also affected by their service age. The following describes the degradation process and state transition of the multi-state components included in the multi-state system. Most scholars use the homogeneous continuous-time Markov to describe this process [6], which means the transformation of a component from one state to another is only related to its current state and is not related to its service age [7]. Considering the influence of service age, in order to more accurately describe the random deterioration rule of multi-state components in the system,the non-homogeneous continuous-time Markov is adopted to describe the relationship between the state transition probability and the service age.

### 3.1. Reliability modeling of multi-state components

In order to describe the deterioration rule of multi-state components more intuitively, Fig. 2 is a diagram of the state transition in the deterioration process of multi-state components.

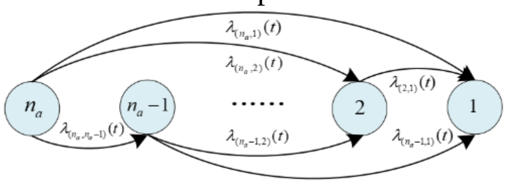


Figure 2: Schematic diagram ofmulti-state component state transitionin the deterioration process

The degradation process of multi-state component $a$ can be described by a stochastic process, which is$\{H_a(t)|t \geq 0\}$. The specific expression is as follows:

$$p_{(a,i)}(t) = P\{H_a(t) = i\}, i = i, \ldots, n_a \tag{5}$$

The influence of the service age is also taken into account,that is, the transition probability of a multi-state component from one state $i$to another state $j$considers the factors related to the service age. The transition probability is assumed to be an increasing function$\lambda_{(i,j)}^{a}(t), (j \in \{i-1, i-2, \ldots, 1\})$ with the independent variable of the service age. Firstly, the non-homogeneous continuous-time Markov is used to obtain the state probability of each multi-state component. Assume its initial condition is the best state with the probability$p_{(a,n_a)} = 1$ and any other state with the probability$p_{(a,i)} = 0, (1 \leq i \leq n_a)$ [8]. Then, the probability that the multi-state component $a$ at the best state $n_a$ can be expressed as follows:

$$p_{(a,n_a)}(t) = \exp\left[-\int_0^t \sum_{i=1}^{n_a-1} \lambda_{(n_a,i)}^{a}(\tau) d\tau\right] \tag{6}$$

Among which, the probability that the multi-state component is at the state $n_a - 1$ is expressed as follows:

$$p_{(a,n_a-1)}(t) = \int_0^t \exp\left[-\int_0^{\tau_1} \sum_{i=1}^{n_a-1} \lambda_{(n_a,i)}^{a}(s) ds\right] \exp\left[-\int_{\tau_1}^{t} \sum_{i=1}^{n_a-2} \lambda_{(n_a-1,i)}^{a}(s) ds\right] \lambda_{(n_a,n_{a-1})}^{l}(\tau_1) d\tau_1 \tag{7}$$

At the same time, the probability sum of each state of the multi-state component must satisfy$\sum_{i=1}^{n_a} p_{(a,i)}(t) = 1$.

Based on these analyses, the probability value of each state of the multi-state component is expressed. Then, the state probability of a multi-state component at any time $t$ can be expressed by a universal generating function. The specific form is as follows:

$$\varphi_a(z,t) = \sum_{i=1}^{n_a} p_{(a,i)}(t) \cdot z^{h_{(a,i)}} = p_{(a,1)}(t) \cdot z^{h_{(a,1)}} + p_{(a,2)}(t) \cdot z^{h_{(a,2)}} + \ldots + p_{(a,n_a)}(t) \cdot z^{h(a,n_a)} \tag{8}$$

Where $Z$ is the z-transformation, which does not participate in the actual calculation.$h_{(a,i)}$represents a variety of states ofthe multi-state component $a$.

### 3.2. Reliability modeling of multi-state systems

Similarly, the system's performance distribution at any time is expressed as follows:

$$\varphi_s(z,t) = \sum_{i=1}^{N_s} p_{(s,i)}(t) \cdot z^{h_{(s,i)}} = p_{(s,1)}(t) \cdot z^{h_{(s,1)}} + p_{(s,2)}(t) \cdot z^{h_{(s,2)}} + \ldots + p_{(s,N_s)}(t) \cdot z^{h_{(s,N_s)}} \tag{9}$$

Where $N_S$ is the number of systemstates,if the multi-state system to be analyzed consists of $Q$ components, and the components are independent of each other. The system expression of the above formula can also be expressed by a combination of multiple components as follows:

$$\begin{aligned}\varphi_s(z,t) &= \Omega_f\{\varphi_1(z,t),\dots,\varphi_Q(z,t)\} \\ &= \Omega_f\left\{\sum_{i_1=1}^{n_1} p_{(1,i_1)}(t)\cdot z^{h_{(1,i_1)}},\dots,\sum_{i_Q=1}^{n_Q} p_{(Q,i_Q)}(t)\cdot z^{h_{(Q,i_Q)}}\right\} \\ &= \sum_{i=1}^{N_s}\left(p_{(s,i)}(t)\cdot z^{h_{(s,i)}}\right)\end{aligned} \tag{10}$$

Where $\Omega_f(\cdot)$ represents the physical configuration function, which reflects is used to describe the system state determined by the state of the system and its components.

The reliability of a multi-state system at time $t$ is defined as the probability that the overall state of the system is greater than or equal to the actual mission requirement limit $w$, which is specifically expressed as follows:

$$R(t,w) = \sum_{i=1}^{N_s} p_{si}(t) 1(h_{si} - w \geq 0) \tag{11}$$

Where $1(\cdot)$ is the indicator function, which meets $1(\text{True}) = 1, 1(\text{False})=0$. It means that the reliability of the multi-state system is obtained by adding up all the state probability values that are not less than the specified requirements of the system.

Then, the specific form of the mean time before failure (MTTF) of the multi-state system is derived as:

$$MTTF = \int_0^{\infty} R(t,w)dt = \int_0^{\infty}\sum_{i=1}^{N_s} p_{si}(t)1(h_{si} - w \geq 0)dt \tag{12}$$

## 4. Maintenance decision optimization of the multi-state system considering imperfect maintenance

Based on the reliability model for multi-state systemsconstructed above, the maintenance decision for the multi-state system is modeled and optimized in this section.

### 4.1. Model assumptions

Before establishing the model of the maintenance decision, it is necessary to predefine the following assumptions about the life cycle maintenance process of the multi-state system:

(1) With the continuous increase in the number of maintenance actions of components, the transitionintensity between each state will increase. That is, the intensity of a single multi-state component from the best state to the poor state is increasing.

(2) The quasi-renewal theory is used to describe the deterioration process of multi-state components, and the maintenance effect is only related to the number of maintenance actions experienced.

(3) A life cycle $\phi_i$ includes $k$-1 imperfect maintenance cycles and one replacement cycle [9]. There is no maintenance delay for maintenance.

(4) The parameters related to system maintenance cost are defined as: $c_w$ is the output profit per unit time, $c_f$ is the maintenance expenditure per unit time, $c_{rw}$ is the replacement expenditure per unit time, $c_r$ is the fixed ordering expenditure. It is usually considered that $c_f$ is much smaller than $c_r$.

### 4.2. Maintenance decision optimization of multi-state systems

Calculate the benefit rate of the multi-state system after ($k$-1)-th imperfect maintenance and one replacement. The specific expression is as follows:

$$Q(K)=\frac{c_w E(S(K))-c_f E(Y(K-1))-c_{rw} E(Z)-c_r}{E(Y(K-1))+E(X(K))+E(Z)} \tag{13}$$

In the above formula, some expressions of the expectations are as follows:

$$E(X(K))=E\left(\sum_{k=1}^{K} X_k^s\right)=\sum_{k=1}^{K} E(X_k^s)=\sum_{k=1}^{K} \mu_{X_k^s} \tag{14}$$

$$E(Y(K-1))=E\left(\sum_{k=1}^{K-1} Y_k^s\right)=\sum_{k=1}^{K-1} E(Y_k^s) \tag{15}$$

$$E(Z)=\mu_Z \tag{16}$$

Besides, the duration of each life cycle of the multi-state system is:

$$\begin{aligned} E(S(K)) &= E\left(\sum_{k=1}^{K}\sum_{i=1}^{N_s} S_{k,i}\cdot h_{si}\cdot 1(h_{si}-w\geq 0)\right) \\ &= \sum_{i=1}^{N_s}\sum_{k=1}^{K} E(S_{k,i})\cdot h_{si}\cdot 1(h_{si}-w\geq 0) \\ &= \sum_{\substack{i=n \\ h_k\geq w}}^{N_s}\sum_{k=1}^{K} \mu_{S_{k,i}}\cdot h_{si} \end{aligned} \tag{17}$$

Finally, the objective function of the maintenance decision is to optimize the number of imperfect maintenance actions *k*-1 in order to make the maintenance benefit achieves the maximum value.

$$K^* = \arg\max(Q_K) \tag{18}$$

## 5. Case study

In this section, theaero-engineis taken as an example and the proposed maintenance decision optimization method is applied to optimize the maintenance decision considering imperfect maintenance on the aero-engine. The main components of the engine include the nozzle (A), compressor (B), and combustion chamber, which consists mainly of the diffuser (C) and annular chamber combustor (D), as shown in Fig. 3.

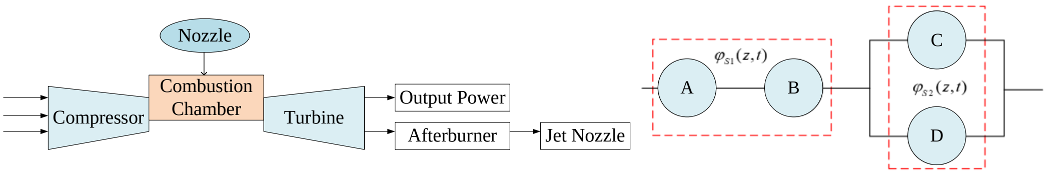


Figure 3: Main components of the aero-engine and its reliability block diagram

### 5.1. Thesystem mission process

All the states of the main components in the first imperfect maintenance cycle and their corresponding state transition intensities are calculated to describe the deterioration process in which the components are affected by the service age. The state transition process of each component analyzed above is depicted in Fig. 4.

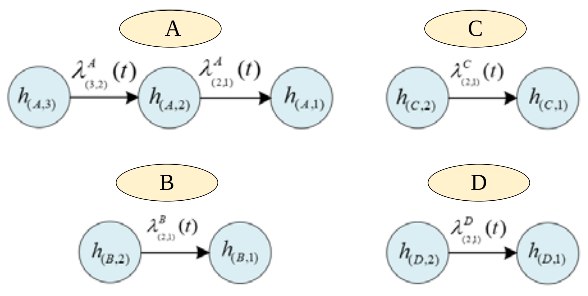

Figure 4: The state transition process of multi-state components

The multiple states,the quasi-renewal processes for each multi-state component, as well as the associated characteristics of imperfect maintenance, are listed in Table 1.

Table 1: Relevant characteristics of the components of a multi-state system.

| Component | A | B | C | D |
|---|---|---|---|---|
| State | $h_{(A,3)}=1.0$ | $h_{(B,2)}=1.0$ | $h_{(C,2)}=0.45$ | $h_{(D,2)}=0.55$ |
| | $h_{(A,2)}=0.6$ | $h_{(B,1)}=0$ | $h_{(C,1)}=0$ | $h_{(D,1)}=0$ |
| | $h_{(A,1)}=0$ | \ | \ | \ |
| State transition intensity( $year^{-1}$ ) | $\lambda_{(3,2)}^{A}(t)$ $=0.6+0.2t$ | $\lambda_{(2,1)}^{B}(t)$ $=0.9+0.2t^2$ | $\lambda_{(2,1)}^{C}(t)$ $=1.1+0.1t^2$ | $\lambda_{(2,1)}^{D}(t)$ $=0.9+0.3t^2$ |
| | $\lambda_{(2,1)}^{A}(t)$ $=1.2+0.1t$ | \ | \ | \ |
| $\alpha$ | 0.9 | 0.8 | 0.9 | 0.7 |
| $\beta$ | 1.2 | 1.1 | 1.1 | 1.3 |

## 5.2. System reliability analysis

To analyze the reliability of a multi-state system, its components need to be discussed first. Since the states of the multi-state components can directly determine the overall state of the system, the probability of each component state can be obtained by Equations (6) and (7), and then put them intoEquation (8) to obtain the universal generating function of each multi-state componentin the first imperfect maintenance cycle. Combined with the corresponding physical configuration function, the state of the entire multi-state system during the first imperfect maintenance cycle can be obtained as follows:

$$\varphi_s^1(z,t)=\sum_{i=1}^{5} p_{si}^1(t)\cdot z^{h_{si}} \tag{19}$$

Table 2 shows the probability of each state of the multi-state system in the first imperfect maintenance cycle. It is assumed that when the overall output state of an aero-engine is greater than or equal to 55% of its optimal state, the specified mission requirements will not be affected, however, when it is less than 55% of its optimum performance, the maintenance personnel will need to take immediate maintenance actions.

Therefore, the reliability of the multi-state system in the first maintenance cycle is expressed as follows:

$$R_s^1(t,w)=\sum_{i=1}^{5} p_{si}^1(t)\cdot 1(h_{si}\geq 0.55)=p_{s3}^1+p_{s4}^1+p_{s5}^1 \tag{20}$$

Table 2: Probability of each state of multi-state systems.

| State | Probability of multi-state systems |
|---|---|
| $h_{s5}=1$ | $p_{s5}^{1}(t)=p_{A,3}\cdot p_{B,2}\cdot p_{C,2}\cdot p_{D,2}$ |
| $h_{s4}=0.6$ | $p_{s4}^{1}(t)=p_{A2}\cdot p_{B2}\cdot p_{C2}\cdot p_{D2}$ |
| $h_{s3}=0.55$ | $p_{s3}^{1}(t)=p_{A3}\cdot p_{B2}\cdot p_{C1}\cdot p_{D2}+p_{A2}\cdot p_{B2}\cdot p_{C1}\cdot p_{D2}$ |
| $h_{s2}=0.45$ | $p_{s2}^{1}(t)=p_{A3}\cdot p_{B2}\cdot p_{C2}\cdot p_{D1}+p_{A2}\cdot p_{B2}\cdot p_{C2}\cdot p_{D1}$ |
| $h_{s1}=0$ | $p_{s1}^{1}(t)=p_{B1}+p_{A1}\cdot p_{B2}+p_{A3}\cdot p_{B2}\cdot p_{C1}\cdot p_{D1}$ $+p_{A2}\cdot p_{B2}\cdot p_{C1}\cdot p_{D1}$ |

## 5.3. Maintenance decision optimization

According to the above discussion, it is required to obtain the duration of states of the entire system and the time spent on imperfect maintenance in order to make a reasonable maintenance decision must. Since the output state greater than or equal to 55% of its best states is regarded as the normal operating state, the state probabilities of the multi-state system under the specified missionrequirement in the *k*-th imperfect maintenance cycle is calculated by Equation (11).

Table 3: Benefit-related parameters for multi-state systems (cost ($\times 10^3$)).

| $c_f$ | $c_r$ | $c_w$ | $c_{rw}$ | $\mu_{Y_1^A}$ | $\mu_{Y_1^B}$ | $\mu_{Y_1^C}$ | $\mu_{Y_1^D}$ | $\mu_Z$ |
|---|---|---|---|---|---|---|---|---|
| 4 | 80 | 160 | 10 | 0.025 | 0.02 | 0.03 | 0.015 | 0.12 |

Benefit-related parameters for multi-state systems are listed in Table 3. Based on the above analysis and related data, it can be calculated that the benefit rate of the multi-state system changes with the number of imperfect maintenance actions in a life cycle.

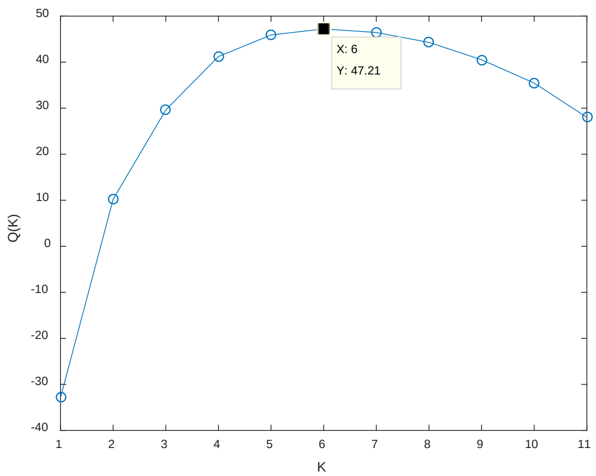


Figure 5: The expected average benefit per unit time of different replacement decisions

From Fig. 5, it can be seen that the optimized maintenance decision based on imperfect maintenance is obtained as replacing the new system after 5 imperfect maintenance actions because it has the maximum expected benefit per unit time as $Q(6)=4.721\times 10^3$.

## 6. Conclusions

This paper analyzes a maintenance decision optimization method of multi-state systems from the perspective of "overall performance". Firstly, an imperfect maintenance model of the multi-state component based on the quasi-renewal theory is given, which is fully considered that the component is not in a new state after maintenance. Then, the relationship between the states of components and system is considered and a non-homogeneous continuous-time Markov model is constructed to realize the quantitative representation of multi-state system reliability. Finally, considering the impact of imperfect maintenance, the optimal maintenance strategy of multi-state system is given by optimizing the number of imperfect maintenance and maximizing the expected benefit per unit time in the whole life cycle of the multi-state system.

Taking an aero-engine as an example, the feasibility and practicability of this method are verified. The results show that this method can not only accurately characterize the degradation state of the multi-state system, but also achieve maintenance optimization. It can guide maintenance personnel to perform scientific maintenance and replacement, and minimize maintenance costs.

## References


*[1] Zhang Y L , Wang G J . An optimal replacement policy for a multistate degenerative simple system [J]. Applied Mathematical Modelling, 2010, 34(12):P.4138-4150.*

*[2] Ning Wang, ShudongSun, Shubin Si. Optimal Predictive Maintenance Policy for Multi-State Deteriorating System under Periodic Inspections [J]. 2010 IEEE: p. 631-637.*

*[3] Chen M , Jiang Y , Zhou D . Decentralized Maintenance for Multistate Systems With Heterogeneous Components[J]. IEEE Transactions on Reliability, 2018:1-14.*

*[4] Attia A F, Elela E D A ,Hosham H A . The Optimal Warranty and Preventive Maintenance Policy for the Four-State System [J]. Isrn Applied Mathematics, 2011:722-736.*

*[5] Zhang L, Zhou Y, Huang C. An approximate hybrid approach to maintenance optimisation for a system with multistate components [J]. International Journal of System Assurance Engineering and Management, 2017, 8(1):189-196.*

*[6] Lisnianski A. Extended block diagram method for a multi-state system reliability assessment [J]. Reliability Engineering & System Safety, 2007, 92 (12): 1601-1607.*

*[7] Trivedi K S. Probability and statistics with reliability queuing, and computer science applications[M]. Probability and statistics with reliability, queuing, and computer science applications, 2019,pp:102-114,.*

*[8] Liu Y W ,Kapur K C . Customer's Cumulative Experience Measures for Reliability of Non-Repairable Aging Multi-State Systems[J]. Quality Technology & Quantitative Management, 2007, 4(2):225-234.*

*[9] Jain M , Shekhar C , Shukla S . A time-shared machine repair problem with mixed spares under N-policy [J]. Journal of Industrial Engineering International, 2016, 12(2) : 145-157 .*