Journal of Network Computing and Applications (2022)
Clausius Scientific Press, Canada

DOI: 10.23977/jnca.2022.070104
ISSN 2371-9214 Vol. 7 Num. 1

The Condition of Reaching the Minimum of Vector-Valued Mapping under Lexicographic Order

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Keywords: Lexicographic order, minimal element, lower-C semicontinuous, properly quasi-convex

Abstract: In this paper, we study the existence of minimal elements of lower-C semicontinuous vector-valued mappings on compact sets and properly quasi-convex vector-valued mappings on compact convex sets in the sense of lexicographic order, the first and second criteria for the existence of minimal elements are obtained. In addition, this paper illustrates the extensive application of the first criterion in the field of discontinuous vector-valued mapping and the stronger applicability of the second criterion compared with the first criterion.

1. Introduction

The minimization of vector-valued mappings under partial order has become a research hotspot in the field of mathematics. Most of the existing research results are related to the closed convex pointed cone. In 1989, F. Ferro[1] proposed that if set Y_0 is a compact convex subset of Hausdorff space Y, C is a closed convex cone in topological vector space V, φ is a lower-C semicontinuous and properly quasi-convex mapping of $Y_0 \to V$, then it exists $y_0 \in Y_0$ such that $\varphi(y_0) = Min(\varphi(Y_0) | C)$. In addition, in 1997, F. Ferro[2] gave relevant conclusions for set-valued mapping: Let Λ be a compact topological vector space and C be a closed convex pointed cone in topological vector space Y, T is a set-valued mapping of $\Lambda \to Y$ satisfying that T is lower-C semicontinuous and any $\lambda \in \Lambda$, $T(\lambda)+C$ is C-complete, then $Min(T(\Lambda) | C) \neq \phi$.

Lexicographic order is the complet order induced by lexicographic cone. The lexicographic cone is not a closed convex pointed cone, and the closure of the lexicographic cone is no longer a pointed cone, so it can't induce partial order. Therefore, the above conclusion is not applicable to the lexicographic cone. There are also relevant conclusions on the minimization of vector-valued mapping under lexicographic order. In 2010, X.B. Li[3] and others proposed that if a set is a compact set, the image set after continuous vector-valued mapping must have minimal elements under lexicographic order. However, there is still no relevant conclusion on the minimization of discontinuous vector-valued mapping under lexicographic order.

In this paper, we first study the minimization of lower-C semicontinuous vector-valued mappings acting on compact sets under lexicographic order, and obtain the first criterion for the existence of minimal elements. The first criterion fills the gap of judging the existence of minimal elements by discontinuous vector valued mapping to a certain extent. In addition, as an improvement of the first criterion, this paper also proposes the second criterion for the existence of minimal elements. The second criterion is not only applicable to the existence of minimal elements for discontinuous vector valued mappings, but also stronger than the first criterion in the application of the category of continuous vector valued mappings.

2. Preparation knowledge

Let E be a topological vector space, and we call the nonempty subset C of E a cone if $\lambda \cdot C \subset C$ for every $\lambda \geq 0$. C is called a convex cone if and only if $C + C \subset C$. In addition, we call C a pointed cone if $C \bigcap (-C) = \{\theta\}$ where θ represents the zero element in E. Throughout this paper, unless specified otherwise, we default that E is the topological vector space and C is a pointed convex cone in E.

Definition 1.1[3]: Let $I_n = \{1, 2, \cdots n\}$, for any $v \in \mathbb{R}^n$ and $i \in I_n$, v_j denote the j-th coordinate of v. The lexicographic cone of $\mathbb{R}^n$ is defined as the set of all vectors whose first nonzero coordinate (if any) is positive:

$$K_{lex} = \{0\} \cup \{v \in \mathbb{R}^n : \exists i \in I_n \; s.t. v_i > 0; Not\ exist\ j \in I_n, j < i \; s.t. v_j \neq 0\}$$

Definition 1.2[4]: The partial order relationship between elements in E is defined as: $y_1 \leq y_2$, if $y_2 - y_1 \in C$. In particular, if C is a lexicographic cone, this partial order relationship is called lexicographic order.

Definition 1.3[5]: Let B be a nonempty subset of E. Say y_0 is the minimal element of B if $B \bigcap (y_0 - C) = \{y_0\}$. Record the set composed of all minimal elements in B as $Min(B|C)$.

In particular, when C is a lexicographic cone, $Min(B|C)$ is a singleton.

Definition 1.4[1]: Let X be a nonempty subset of E, vector-valued mapping $f: X \to Z$ is called lower-C semicontinuous if set $f^{-1}(z - C) = \{x \in X : f(x) \in z - C\}$ is closed in X for any $z \in Z$.

Definition 1.5[6]: Let X and Y be topological spaces, the mapping $f: X \to Y$ is said to be continuous at point $x \in X$, if for any open set U containing point f(x) in Y, it exists open set V in X s.t. $x \in V$ and $f(V) \subset U$. If f is continuous at every point of topological space X, then f is called a continuous mapping.

Definition 1.6[7]: A set $A \subseteq E$ is said to be strongly C-complete, if it has no covers of the form $\{(x_\alpha - C)^c : \alpha \in I\}$, with $\{x_\alpha\}$ being a decreasing net in A.

Definition 1.7[8]: Let $A \subseteq E$ and $x \in E$. The set $A \bigcap (x - C)$ is called a section of A at x.

Definition 1.8[9]: Let Y be a convex set in E, We say that $\varphi: Y \to R^n$ is properly quasi-convex, if for every $y_1, y_2 \in Y$, and $t \in [0,1]$, we have

$$\varphi(ty_1 + (1-t)y_2) \leq \varphi(y_1), \text{ or } \varphi(ty_1 + (1-t)y_2) \leq \varphi(y_2).$$

For the convenience of narration, this paper records the lexicographic cone as K_{lex}, the cone hull of X is recorded as $cone(X)$, the convex hull of X is recorded as $conv\{X\}$, the boundary of X is recorded as $\partial(X)$.

3. The first criterion for the existence of minimal elements

Lemma 2.1[10]: Let E be Hausdorff topological vector space, A is a nonempty subset in E, C is a pointed convex cone in E. $Min(A|C)$ is nonempty if and only if A has a nonempty strongly C-complete section.

Obviously, if A is strongly C-complete, then $Min(A|C) \neq \phi$. This is because if we take any $m \in E$, it is easy to know that $A \bigcap (m - C)$ is strongly C-complete.

Theorem 2.1: Let X be Hausdorff topological vector space, $X_0 \subseteq X$ and X_0 is a compact set. If $f: X \to R^n$ is lower-K_{lex} semicontinuous, then $Min(f(X_0)|K_{lex}) \neq \phi$.

Proof: By Lemma 2.1, it only prove that $f(X_0)$ is strongly $K_{lex}-$complete. By the contrary, assume that $f(X_0)$ is not strongly $K_{lex}-$ complete, then $f(X_0)$ has a coverage of the form $\{(f(x_\alpha - K_{lex})^c : \alpha \in I\}$, where $\{f(x_\alpha)\}$ is a decreasing net in $f(X_0)$. Then

$$f(X_0) \subseteq \bigcup_{\alpha \in I} (f(x_\alpha) - K_{lex})^c,$$

Therefore

$$X_0 \subseteq f^{-1}[\bigcup_{\alpha \in I} (f(x_\alpha) - K_{lex})^c] = \bigcup_{\alpha \in I} f^{-1}[(f(x_\alpha) - K_{lex})^c] = \bigcup_{\alpha \in I} [f^{-1}(f(x_\alpha) - K_{lex})]^c$$

Since f is lower-K_{lex} semicontinuous, then for $\forall \alpha \in I$, $f^{-1}[f(x_\alpha) - K_{lex}]$ is closed in X_0. So $\bigcup_{\alpha \in I} [f^{-1}(f(x_\alpha) - K_{lex})]^c$ is a open coverage of X_0. Since X_0 is a compact set, then it exists

$$\{\alpha_1, \alpha_2 \cdots \alpha_n\} \subseteq I \text{ s.t. } X_0 \subseteq \bigcup_{i=1}^{n} [f^{-1}(f(x_{\alpha_i}) - K_{lex})]^c.$$

Then

$$f(X_0) \subseteq f\{\bigcup_{i=1}^{n} [f^{-1}(f(x_{\alpha_i}) - K_{lex})]^c\} = \bigcup_{i=1}^{n} f\{[f^{-1}(f(x_{\alpha_i}) - K_{lex})]^c\}$$

$$= \bigcup_{i=1}^{n} f[f^{-1}((f(x_{\alpha_i}) - K_{lex})^c)] = \bigcup_{i=1}^{n} (f(x_{\alpha_i}) - K_{lex})^c.$$

This is impossible by the contradiction assumption which implied $\{(f(x_\alpha - K_{lex})^c\}$ is a coverage of $f(X_0)$.Therefore $f(X_0)$ is strongly $K_{lex}-$complete, and $Min(f(X_0)|K_{lex}) \neq \phi$.

In reference [3], X.B. Li[3] and others put forward relevant conclusions on the existence of minimal elements under lexicographic order for continuous vector-valued mappings acting on compact sets. However, when vector-valued mappings are discontinuous mappings, this conclusion cannot be used to judge the existence of minimal elements. As illustrated in example 2.2 below, the first criterion for the existence of minimal elements fills the gap in judging the existence of minimal elements when the mapping is discontinuous to a certain degree.

Example 2.2: let $X_0 = \{(x, y) | -1 \le x \le 1, -1 \le y \le 1\}$, $f: R^2 \to R^2$ satisfies

$$f(x, y) = \left\{ (z_1, z_2) \middle| \; z_1 = x, z_2 = \begin{cases} 1, & -1 \le x \le 0 \\ 2, & 0 < x \le 1 \end{cases} \right\}.$$

Obviously, X_0 is a compact set in R^2 and f is not continuous. Since any $z = (z_1, z_2) \in f(X_0)$,

$$f^{-1}(z - K_{lex}) = \{(x, y) | -1 \le x \le z_1, -1 \le y \le 1\}$$

is closed in X_0, so f is $K_{lex}-$complete. In fact, it is not difficult to find that (-1,1) is the minimal element in $f(X_0)$.

However, the first criterion also has its limitations. The following example illustrate that continuous mapping is not necessarily $K_{lex}-$lower semicontinuous mapping.

Example 2.3: Change the mapping f in example 2.2 to $f(x,y)=(x,y)$, other conditions remain unchanged.

Obviously, f is a continuous mapping. Consider $z_0=(1,-1)$, since

$$f^{-1}(z_0-K_{lex})=\{(x,y)\mid -1\le x<1,-1\le y\le 1\}\bigcup\{z_0\}$$

is not closed in X_0, therefore f is not lower-K_{lex} semicontinuous.

4. The second criterion for the existence of minimal elements

Lemma 3.1[1]: If set X_0 is a compact convex subset of Hausdorff topological vector space X, C is a closed convex cone in topological vector space Z. If $f:X_0\to Z$ is properly quasi-convex and lower-C semicontinuous, then it exists $x_0\in X_0$ s.t. $\{f(x_0)\}=Min(f(X_0)\mid C)$.

Theorem 3.1: Let Y is a Hausdorff topological vector space, and Y_0 is a compact convex subset of Y. $f:Y\to R^2$ is a vector valued mapping, if f satisfies:

(1) It exists $x_0<0$, corresponding a closed convex pointed cone $C_0=cone\{conv\{(0,1),(1,x_0)\}\}$ s.t. f is properly quasi-convex for C_0.

(2) For any $n\in N_+$, $C_n=cone\{conv\{(0,1),(1,x_0-n)\}\}$, f is lower-C_n semicontinuous.

Then $Min(f(Y_0)\mid K_{lex})\ne\phi$.

Proof: According to the definition of C_n,it is easy to know that $C_n\subseteq C_{n+1}$.Since f is properly quasi -convex for C_0, f is properly quasi-convex for $C_n,\forall n\in N$. According to Lemma 3.1, for any $n\in N$, There exist $y_n\in f(Y_0)$ s.t. $\{y_n\}=Min(f(Y_0)\mid C_n)$.

Now we prove that for any $n\in N$, $Min(f(Y_0)\mid C_n)=\{y_0\}$.

Since $Min(f(Y_0)\mid C_{n+1})=\{y_{n+1}\}$,we have $f(Y_0)\bigcap(y_{n+1}-C_{n+1})=\{y_{n+1}\}$. Since $C_{n+1}\supseteq C_n$,we have $f(Y_0)\bigcap(y_{n+1}-C_{n+1})\supseteq f(Y_0)\bigcap(y_{n+1}-C_n)$ and $f(Y_0)\bigcap(y_{n+1}-C_n)\subseteq\{y_{n+1}\}$. It is obvious that $\{y_{n+1}\}\subseteq f(Y_0)\bigcap(y_{n+1}-C_n)$, so $f(Y_0)\bigcap(y_{n+1}-C_n)=\{y_{n+1}\}$, i.e. $y_{n+1}\in Min(f(Y_0)\mid C_n)=\{y_n\}$.

Thus, for any $n\in N$,

$$Min(f(Y_0)\mid C_n)=\{y_0\} \tag{1}$$

Next we prove that $y_0\in Min(f(Y_0)\mid K_{lex})$. Since $y_0\in f(Y_0)\bigcap(y_0-K_{lex})$,it only need to prove that for any $y\in f(Y_0), y\ne y_0$,we have $y\notin(y_0-K_{lex})$. By the contrary, assume exist $m\in f(Y_0)\ and\ m\ne y_0$ has $y\in(y_0-K_{lex})$. Record the coordinate of y_0 as (p_1,q_1),the coordinate of m as (p_2,q_2).

① If $m\in(y_0-K_{lex})\bigcap\partial(y_0-K_{lex})$, for any $n_1\in N_+$, we have $m\in f(Y_0)\bigcap(y_0-C_{n_1})$. Thus $f(Y_0)\bigcap(y_0-C_{n_1})\ne\{y_0\}$. This is contradictory to (1).

② If $m \in (y_0 - K_{lex}) \setminus \partial(y_0 - K_{lex})$, it exists $n_2 = [|\frac{q_2 - q_1}{p_2 - p_1}| - x_0] + 1$ s.t. $m \in f(Y_0) \bigcap (y_0 - C_{n_2})$. This is contradictory to (1).

According to the combination of ① and ②, the hypothesis is not tenable. Thus $y_0 \in Min(f(Y_0) | K_{lex})$ i.e. $Min(f(X_0) | K_{lex}) \neq \phi$.

Example 3.2: Let $X_0 = \{(x, y) | 1 \leq x \leq 2, 1 \leq y \leq 2\}$, $f : R^2 \to R^2$ satisfies

$$f(x, y) = (x + y, -x - y),$$

Then X_0 is a compact convex set in R^2 and f satisfies:

(1) Exist $C_1 = cone\{conv\{(0,1), (1,-1)\}\}$ s.t. f is properly quasi-convex for C_1.

(2) For any $n \geq 2$, $C_n = cone\{conv\{(0,1), (1,-n-1)\}\}$, since f is continuous mapping, f is lower-C_n semicontinuous.

It is not difficult to find that (2,-2) is the minimal element of $f(X_0)$.

5. Summary

As can be seen from example 2.2, the first criterion for the existence of minimal elements fills the gap in judging the existence of minimal elements when the mapping is discontinuous to a certain extent. However, the first criterion also has its limitations. It is not difficult to find from example 2.3 that some well-known continuous mappings (such as identity mappings) are not lower-K_{lex} semicontinuous. In view of this, the second criterion for the existence of minimal elements is proposed in this paper. The second criterion is also applicable to some problems of judging the existence of minimal elements for discontinuous mappings. For example, example 2.2 also meets the conditions of the second criterion. In addition, since continuous and properly quasi-convex for K_{lex} mappings must meet the conditions of the second criterion, the second criterion is stronger than the first criterion in application.

Acknowledgements

I would like to thank Dr. Caiyun Jin for her guidance on the paper, and also thank National Undergraduate Student Innovation & Entrepreneurship Program for its financial support for the project research.

Accepted another source of funding of National Undergraduate Student Innovation & Entrepreneurship Program (GJDC-2021-01-34)

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