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Application of Mobius inversion in combinatorial problems

Zehan Lin

Guangzhou University (Guangzhou, Guangdong 510006)

Kurisu19990712@163.com

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Abstract: Mobius inversion plays an very important part in number theory mathematics and can be used to solve many combinatorial problems.For some functions f(n), if it is difficult to find its value directly, but it is easy to find the sum of its multiples or divisors as g(n), then the calculation can be simplified through Mobius inversion to obtain the value of $f(n)$.In this article, we provide a method to use Mobius inversion to solve some combinatorial problems efficiently with computer calculation.

1. Introduction

1.1 Mobius Inversion

Theorem 1. For every natural number n > 1, if n isn't prime number, then n can be uniquely decomposed into a product of finite prime numbers

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}, where\ p_1 < p_2 < \ldots < p_k\ are\ all\ prime\ numbers.$$

Let the greatest common divisor be abbreviated by gcd, for Arithmetic function, it satisfies $f(mn) = f(m)f(n)$.We can use Theorem 1 to express the equation as

$$\theta(n) = \theta(p_1^{\alpha_1})\theta(p_2^{\alpha_2})\ldots\theta(p_k)^{\alpha_k}$$

There is a traditional expression of Mobius inversion.

$$if\ F(n) = \sum_{d|n} f(d), then\ f(n) = \sum_{d|n} \mu(d) F(\frac{n}{d})$$

1.2 Mobius function

If $\mu(n)$ is the Mobius function of n, then

$$\mu(n) = \begin{cases} 1, & n = 1 \\ (-1)^k, & n = p_1 p_2 \ldots p_k \\ 0, else \end{cases}$$

Theorem 2. Mobius function is productivity function, which means that $\mu(ab) = \mu(a)\mu(b)$ if it satisfies $gcd(a, b) = 1$.

Theorem 3. For arbitrary positive integer n, it satisfies that

$$\sum_{d|n} \mu\,(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$$

1.3 Number theory block

Theorem 4. $\forall n \in N_+, |\{\lfloor \frac{n}{d} \rfloor | d \in N_+, d <= n\}| <= \lfloor 2\sqrt{n} \rfloor$

Therefore, we can calculate $\sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor$ in $O(\sqrt{n})$ by the code below.

Code for number theory block

```
set i <- 1;
set ans <- 0;
while i <= n
    set j <- n / (n / i);
   set ans <- ans + (j - i + 1) * (n / i);
    set i <- j + 1;
end while
return ans;
```

1.4 Euler sieve for Mobius function

It's clear that we can calculate $\mu(i)(1 <= i <= n)$ based on the information mentioned above in $O(n)$.

Code for Euler sieve

```
set mu[1] <- 1; // mu[i] is the ith Mobius function
set i <- 2;
set tot <- 1;
while i <= n do // n is the upper bound
    if flag[i] == 0 // flag[i] show whether it has been searched
        set tot <- tot + 1;
        set p[tot] <- i; // p[i] is the ith prime number
        set mu[i] <- -1;
    end if
    set j <- 1;
    while j <= tot and i * p[j] <= n do
        set flag[i * p[j]] <- 1;
        if i % p[j] == 0
              mu[i * p[j]] <- 0;
              break;
        end if
    end while
end while
```

2. Application of Mobius inversion

In this section, we will try to apply Mobius invers to a kind of combinatorial mathematics. After that, we are able to find why Mobius inversion plays an important role in number theory.

Let us pay attention to the problem described as "Given n and m, define d(x) as the number of divisors of x, calculate $\sum_{i=1}^{n}\sum_{j=1}^{m} d\,(i,j)$."

$$\sum_{i=1}^{n}\sum_{j=1}^{m} d\,(i,j) = \sum_{x|i}\sum_{y|j}[\,gcd(x,y) = 1]$$

$$F(n) = \sum_{n|d} f\,(d)$$

Here we can make use of Mobius inversion to transfer the equation.

$$f(n) = \sum_{n|d} \mu\,(\lfloor\frac{d}{n}\rfloor)F(d)$$

Therefore, what we need to calculate is

$$ans = \sum_{i=1}^{n}\sum_{j=1}^{m} d\,(i,j) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{x|i}\sum_{y|j}[\,gcd(x,y) = 1] = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{x|i}\sum_{y|j}\sum_{d|gcd(x,y)} \mu\,(d)$$

Then, we can enumerate d and transfer the equation

$$ans = \sum_{d=1}^{min(m,n)} \mu\,(d)\sum_{x=1}^{n}\sum_{j=1}^{m}\sum_{x|i}\sum_{y|j}[\,d|gcd(x,y)]$$

After that, we can enumerate the divisors of i,j and transfer the equation

$$ans = \sum_{d=1}^{min(n,m)} \mu\,(d)\sum_{x=1}^{n}\sum_{y=1}^{m}[\,d|gcd(x,y)]\lfloor\frac{n}{x}\rfloor$$

Finally, we change the enumeration from x, y to dx, dy, which helps us to ignore $[d|gcd(x,y)]$, and get the equation

$$ans = \sum_{d=1}^{min(n,m)} \mu\,(d)\sum_{x=1}^{\lfloor\frac{n}{d}\rfloor}\lfloor\frac{n}{dx}\rfloor\sum_{y=1}^{\lfloor\frac{m}{d}\rfloor}\lfloor\frac{m}{dy}\rfloor$$

It’s clear that we can calculate the equation in $O(\sqrt{n})$ with the help of number theory block.

3. Conclusion

In sum, it’s convenient and efficient for us to solve some combinatorics problems with the use of Mobius inversion. As a matter of fact,Mobius inversion can not only be used in combinatorics problems, but also be used in others fields such as decision theory and physical problems .There is no doubt that we should keep making research on it.

References

[1] Chateauneuf A, Jaffray J Y. Some Characterizations of Lower Probabilities and Other Monotone Capacities through the Use of Mobius Inversion [J]. Mathematical Social ences, 1989, 17 (3): 263-283.

[2] Chen N X, Li M, Liu S J. PHONON DISPERSIONS AND ELASTIC-CONSTANTS OF NI3AL AND MOBIUS-INVERSION [J]. Physics Letters A, 1994, 195 (2): 135-143.
[3] FUJIMOTO, K. Some Characterization of the Systems Represented by Choquet and Multi-Linear Functionals throught the Use of Mobius Inversion [J]. International Journal of Fuzziness and Knowledge-based Systems, 1997, 5.
[4] Liu S J, Li M, Chen N X. Mobius transform and inversion from cohesion to elastic constants [J]. Journal of Physics Condensed Matter, 1993, 5 (26): 4381.
[5] Krot E. A note on mobiusien function and mobiusien inversion formula of fibonacci cobweb poset [J]. Mathematics, 2004, 44 (44): 39-44.
[6] Bayad, Abdelmejid, Navas. Mobius inversion formulas related to the Fourier expansions of two-dimensional Apostol-Bernoulli polynomials [J]. Journal of Number Theory, 2016.