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Robust optimization of generalized countable compact spaces and its application in project management

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Abstract: We first give alternative expressions of some generalized countably compact spaces such as quasi- spaces, quasi-Nagata spaces, M#-spaces and wM-spaces with g-functions. Then by means of these expressions, we present some characterizations of the corresponding spaces with real-valued functions.

1. Introduction

Throughout, a space always means a Hausdorff topological space unless otherwise stated. Let X be a space. Denote by CX (SX) the family of all compact (sequentially compact) subsets of X. τ and τ c denote the topology of X and the families of all closed subsets of X, respectively [1]. F0(X) denotes the family of all decreasing sequences of closed subsets of X with empty intersection. The set of all positive integers is denoted by N while □xn□ denotes a sequence. A real-valued function f on a space X is called lower (upper) semi-continuous [1] if for any real number r, the set $\{x \in X : f(x) > r\}$ $(\{x \in X : f(x) < r\})$ is open. We write L(X) (U(X)) for the set of all lower (upper) semi-continuous functions from X into the unit interval [0, 1]. A g-function for a space X is a map $g : N \times X \to \tau$ such that for each $x \in X$ and $n \in N$, $x \in g(n, x)$ and $g(n + 1, x) \subset g(n, x)$. For a subset $A \subset X$, let $g(n,A) = \cup \{g(n, x) : x \in A\}$. Consider the following conditions.

(q) If $xn \in g(n, x)$ for all $n \in N$, then □xn□ has a cluster point. (quasi-γ) If $xn \in g(n, yn)$ for all $n \in N$ and $yn \to x$, then □xn□ has a cluster point[2].

(β) If $x \in g(n, xn)$ for all $n \in N$, then □xn□ has a cluster point. (quasi-Nagata) If $yn \in g(n, xn)$ for all $n \in N$ and $yn \to x$, then □xn□ has a cluster point. (kβ) For each $K \in CX$, if $K \cap g(n, xn) \neq \emptyset$ for all $n \in N$, then □xn□ has a cluster point. (wN) If $g(n, x) \cap g(n, xn) \neq \emptyset$ for all $n \in N$, then □xn□ has a cluster point. A space that has a g-function satisfying condition (q) ((quasi-γ), (β), (quasi-Nagata), (kβ), (wN)) is called a q-space [2] (quasi-γ space , β-space [4], quasi-Nagata space, kβ-space, wN-space). The g-function satisfying condition (q) is called a q-function. The others are defined analogously. β-spaces were also called monotonically countably metacompact spaces in . kβ-spaces were also called monotonically countably mesocompact spacesand k- MCM spaces.

It is known that a space X is countably compact if and only if every sequence in X has a cluster point. Thus for a countably compact space X, if we let $g(n, x) = X$ for each $x \in X$ and $n \in N$, then we get a g-function for X which clearly satisfies all the conditions listed above. Thus all these spaces can be viewed as generalizations of countably compact spaces. On the other hand, they are also natural generalizations of some corresponding generalized metric spaces. Actually, if we replace '$\langle x_n \rangle$ has a cluster point' in condition (q) ((quasi-γ), (β), (quasi-Nagata), (wN)) with 'x is a cluster point of $\langle x_n \rangle$', then we get the g-function for first countable spaces (γ-spaces, semi-stratifiable spaces, k-semi-stratifiable spaces, Nagata-spaces). In [11], it was shown that most of generalized metric spaces such as γ-spaces, Nagata-spaces, semi-metrizable spaces and quasi-metrizable spaces can be characterized with real-valued functions. A natural question is that, as generalizations of the corresponding generalized metric spaces, whether the generalized countably compact spaces mentioned above can also be characterized with real-valued functions. With the question in mind, in this paper, we shall show that many classes of generalized countably compact spaces such as the spaces mentioned above as well as M#-spaces, wM-spaces can be characterized analogously to the corresponding generalized metric spaces.

2. Alternative expressions of some corresponding spaces

In this section, we give alternative expressions of some corresponding spaces with g-functions which will be used in Section 3.

Lemma 2.1 If $\langle F_n \rangle \in F_0(X)$ and $x_n \in F_n$ for each $n \in N$, then $\langle x_n \rangle$ has no cluster point.

Proof Since $\langle F_n \rangle$ is decreasing and $x_n \in F_n$, we have that $\{x_m : m \geqslant n\} \subset F_n$ for each $n \in N$. Thus $\overline{\{x_m : m \geqslant n\}} \subset F_n$ because F_n is closed. It follows that $\bigcap_{n\in N} \overline{\{x_m : m \geqslant n\}} \subset$ [3] $\bigcap_{n\in N} F_n = \emptyset$. This implies that $\langle x_n \rangle$ has no cluster point.

Proposition 2.2 g is a q-function for a space X if and only if for each $\langle F_n \rangle \in F_0(X)$ and $x \in X$, $F_n \cap g(n, x) = \emptyset$ for some $n \in N$.

Proof Let g be a q-function for X, $\langle F_n \rangle \in F_0(X)$ and $x \in X$. Assume that $F_n \cap g(n, x) \neq \emptyset$ for each $n \in N$ and choose $x_n \in F_n \cap g(n, x)$. Since g is a q-function, $\langle x_n \rangle$ has a cluster point, a contradiction to Lemma 2.1. Conversely, suppose that $x_n \in g(n, x)$ and let $F_n = \overline{\{x_m : m \geqslant n\}}$ for each $n \in N$. Then $F_n \cap g(n, x) \neq \emptyset$ for each $n \in N$. By the condition, $\bigcap_{n\in N} F_n \neq \emptyset$ which implies that $\langle x_n \rangle$ has a cluster point. Therefore, g is a q-function

Proposition 2.3 For a space X, the following are equivalent. (a) g is a quasi-γ function for X. (b) For each $S \in SX$, if $x_n \in g(n, S)$ for each $n \in N$, then $\langle x_n \rangle$ has a cluster point. (c) For each $S \in SX$ and $\langle F_n \rangle \in F_0(X)$, $F_n \cap g(n, S) = \emptyset$ for some $n \in N$.

Proof (a) ⇒ (b). Let g be a quasi-γ function for X and $S \in SX$. Suppose that $x_n \in g(n, S)$ for each $n \in N$. Then there exists $y_n \in S$ such that $x_n \in g(n, y_n)$ for each $n \in N$. Since $S \in SX$, $\langle y_n \rangle$ has a convergent subsequence $\langle y_{n_k} \rangle$ which clearly also converges in X. Since $x_{n_k} \in g(k, y_{n_k})$ and g is a quasi-γ function, $\langle x_{n_k} \rangle$ has a cluster point which is clearly also a cluster point of $\langle x_n \rangle$.

(b) ⇒ (c). Let g be the g-function in (b) and $\langle F_n \rangle \in F_0(X)$. Let $S \in SX$ and suppose that $F_n \cap g(n, S) \neq \emptyset$ for each $n \in N$. Choose $x_n \in F_n \cap g(n, S)$ for each $n \in N$. By (b), $\langle x_n \rangle$ has a cluster point, a contradiction to Lemma 2.1.

(c) ⇒ (a). Let g be the g-function in (c). Suppose that $x_n \in g(n, y_n)$ for all $n \in N$ and $y_n \to x$. Let $S = \{y_n : n \in N\} \cup \{x\}$ and let $F_n = \overline{\{x_m : m \geqslant n\}}$ for each $n \in N$. Then $S \in SX$ and $F_n \cap g(n, S) \neq \emptyset$ for each $n \in N$. By (c), $\bigcap_{n\in N} F_n \neq \emptyset$ which implies that $\langle x_n \rangle$ has a cluster point. Therefore, g is a quasi-γ function.

Proposition 2.4 g is a β-function for a space X if and only if for each $\langle F_n \rangle \in F_0(X)$ and $x \in X$, $x \notin g(n, F_n)$ for some $n \in N$.

Proof Similar to the proof of Proposition 2.2.

Proposition 2.5 For a space X, the following are equivalent.

(a) g is a quasi-Nagata function for X.

(b) For each $S \in S_X$, if $S \cap g(n, x_n) \neq \emptyset$ for each $n \in N$, then $\langle x_n \rangle$ has a cluster point.

(c) For each $S \in S_X$ and $\langle F_n \rangle \in F_0(X)$, $S \cap g(n, F_n) = \emptyset$ for some $n \in N$.

Proof Similar to the proof of Proposition 2.3.

Since $k\beta$-function can be obtained by replacing $S \in S_X$ in (b) of Proposition 2.5 with $K \in C_X$, with a similar argument, we have the following.

Proposition 2.6 g is a $k\beta$-function for a space X if and only if for each $K \in C_X$ and $\langle F_n \rangle \in F_0(X)$, $K \cap g(n, F_n) = \emptyset$ for some $n \in N$.

A space X is called an $M^\#$-space [12] if there exists a sequence $\{F_n\}_{n \in N}$ of closure preserving closed covers of X such that if $x_n \in st(x, F_n)$ for each $n \in N$, then $\langle x_n \rangle$ has a cluster point.

Proposition 2.7 For a space X, the following are equivalent.

(a) X is an $M^\#$-space.

(b) There exists a g-function g for X such that (1) if $g(n, x) \cap g(n, x_n) \neq \emptyset$ for all $n \in N$, then $\langle x_n \rangle$ has a cluster point; (2) if $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$.

(c) There exists a g-function g for X such that (1) for each $\langle F_n \rangle \in F_0(X)$ and $x \in X$, $g(n, x) \cap g(n, F_n) = \emptyset$ for some $n \in N$; (2) if $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$.

Proof (a) $\Rightarrow$ (b). Let $\{F_n\}_{n \in N}$ be a sequence of closure preserving closed covers of X satisfying the condition of an $M^\#$-space. For each $x \in X$ and $n \in N$, put $h(n, x) = X \setminus \cup\{F \in F_n : x \notin F\}$ and $g(n, x) = \cap_{i \leqslant n} h(i, x)$. Then g is a g-function for X and it is clear that g satisfies (2). Suppose that $y_n \in g(n, x) \cap g(n, x_n) \subset h(n, x) \cap h(n, x_n)$ for each $n \in N$. Since F_n covers X, there is $F_n \in F_n$ such that $y_n \in F_n$. Thus $x, x_n \in F_n$ from which it follows that $x_n \in st(x, F_n)$ for each $n \in N$. Therefore, $\langle x_n \rangle$ has a cluster point.

(b) $\Rightarrow$ (c). Let g be the g-function in (b), $\langle F_n \rangle \in F_0(X)$ and $x \in X$. Assume that $g(n, x) \cap g(n, F_n) \neq \emptyset$ for each $n \in N$. Then there exists $x_n \in F_n$ such that $g(n, x) \cap g(n, x_n) \neq \emptyset$ for each $n \in N$. By (1) of (b), $\langle x_n \rangle$ has a cluster point, a contradiction to Lemma 2.1.

(c) $\Rightarrow$ (b). Let g be the g-function in (c). Suppose that $g(n, x) \cap g(n, x_n) \neq \emptyset$ for all $n \in N$. Let $F_n = \{x_m : m \geqslant n\}$ for each $n \in N$. Then $g(n, x) \cap g(n, F_n) \neq \emptyset$ for each $n \in N$. Thus $\cap_{n \in N} F_n \neq \emptyset$ which implies that $\langle x_n \rangle$ has a cluster point.

(b) $\Rightarrow$ (a). Let g be the g-function in (b). For each $x \in X$ and $n \in N$, let $G_n(x) = \cup\{g(n, y) : y \in X, x \notin g(n, y)\}$. For each $n \in N$, let $F_n = \{X \setminus G_n(x) : x \in X\}$. Then F_n is a closed cover of X.

To show that F_n is closure preserving, let $A \subset X$. We show that $\cap\{G_n(x) : x \in A\}$ is open. Let $y \in \cap\{G_n(x) : x \in A\}$. Then for each $x \in A$, $y \in G_n(x)$ and thus there exists $z \in X$ such that $y \in g(n, z)$ and $x \notin g(n, z)$. By (2) of (b), we have that $g(n, y) \subset g(n, z)$ and thus $x \notin g(n, y)$. This implies that $g(n, y) \subset G_n(x)$ and thus $g(n, y) \subset \cap\{G_n(x) : x \in A\}$. It follows that $\cap\{G_n(x) : x \in A\}$ is open. Therefore $\cup\{X \setminus G_n(x) : x \in A\}$ is closed which implies that F_n is closure preserving.

Now, suppose that $x_n \in st(x, F_n)$ for each $n \in N$. Then there exists $y_n \in X$ such that $x_n, x \in X \setminus G_n(y_n)$. Thus for each $y \in X$, if $y_n \notin g(n, y)$ then $x_n, x \notin g(n, y)$. It follows that $y_n \in g(n, x_n)$ and $y_n \in g(n, x)$ and thus $g(n, x) \cap g(n, x_n) \neq \emptyset$. By (1) of (b), $\langle x_n \rangle$ has a cluster point. Therefore X is an $M^\#$-space.

A cover $\mathcal{P}$ of a space X is called a quasi-(mod k)-network [13] if there is a closed cover $\mathcal{H}$ of X by countably compact subsets such that whenever $H \subset U$ with $H \in \mathcal{H}$ and $U \in \tau$, then $H \subset P \subset U$ for some $P \in \mathcal{P}$. X is called a $\Sigma^{\#}$-space [13] if it has a σ-closure-preserving closed quasi-(mod k)-network.

Lemma 2.8 X is a $\Sigma^{\#}$-space if and only if there exists a g-function g for X such that (1) if $x \in g(n, x_n)$ for all $n \in N$, then $\langle x_n \rangle$ has a cluster point; (2) if $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$. The g-function in the above lemma is called a $\Sigma^{\#}$-function. We see that a $\Sigma^{\#}$-function is precisely a β-function which satisfies an additional condition. Thus by Proposition 2.4, we have the following.

Proposition 2.9 g is a $\Sigma^{\#}$-function for X if and only if (1) for each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, $x \notin g(n, F_n)$ for some $n \in N$; (2) $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$.

A space X is called a wM-space [15] if there exists a sequence $\{\mathcal{G}_n\}_{n\in N}$ of open covers of X such that if $x_n \in st^2(x, \mathcal{G}_n)$ for each $n \in N$, then $\langle x_n \rangle$ has a cluster point. Notice that without loss of generality, we may assume that $\mathcal{G}_{n+1} \prec \mathcal{G}_n$ for each $n \in N$.

Proposition 2.10 For a space X, the following are equivalent.

(a) X is a wM-space.

(b) There exists a g-function g for X such that (1) if $g(n, x)\cap g(n, x_n) \neq \emptyset$ for all $n \in N$, then $\langle x_n \rangle$ has a cluster point; (2) for each $x, y \in X$ and $n \in N$, $y \in g(n, x)$ if and only if $x \in g(n, y)$.

(c) There exists a g-function g for X such that (1) for each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, $g(n, x) \cap g(n, F_n) = \emptyset$ for some $n \in N$; (2) for each $x, y \in X$ and $n \in N$, $y \in g(n, x)$ if and only if $x \in g(n, y)$.

(d) There exists a g-function g for X such that (1) for each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, $x \notin g(n, F_n)$ for some $n \in N$; (2) for each $x, y \in X$ and $n \in N$, $y \in g(n, x)$ if and only if $x \in g(n, y)$.

Proof (a) $\Rightarrow$ (b). Let $\{\mathcal{G}_n\}_{n\in N}$ be a sequence of open covers of X satisfying the condition of a wM-space and $\mathcal{G}_{n+1} \prec \mathcal{G}_n$ for each $n \in N$. For each $x \in X$ and $n \in N$, let $g(n, x) = st(x, \mathcal{G}_n)$. Then g is a g-function for X and it is clear that g satisfies (2). Suppose that $g(n, x)\cap g(n, x_n) \neq \emptyset$ for each $n \in N$. Then $x_n \in st^2(x, \mathcal{G}_n)$ for each $n \in N$ and thus $\langle x_n \rangle$ has a cluster point.

(b) $\Rightarrow$ (c). is similar to the proof of (b) $\Rightarrow$ (c) of Proposition 2.7.

(c) $\Rightarrow$ (d). is clear.

(d) $\Rightarrow$ (a). Let g be the g-function in (d). For each $n \in N$, let $\mathcal{G}_n = \{g(n, x), x \in X\}$. Then $\{\mathcal{G}_n\}_{n\in N}$ is a sequences of open covers of X.

Claim 1 If $x_n \in g(n, x)$ for all $n \in N$, then $\langle x_n \rangle$ has a cluster point.

Proof of Claim 1 For each $n \in N$, let $F_n = \{x_m : m \geqslant n\}$. Assume that $\langle x_n \rangle$ has no cluster point. Then $\langle F_n \rangle \in \mathcal{F}_0(X)$. By (1), $x \notin g(k, F_k) \supset g(k, x_k)$ for some $k \in N$. By (2), $x_k \notin g(k, x)$, a contradiction.

Claim 2 If $g(n, x) \cap g(n, x_n) \neq \emptyset$ for all $n \in N$, then $\langle x_n \rangle$ has a cluster point.

Proof of Claim 2 Choose $y_n \in g(n, x) \cap g(n, x_n)$ for each $n \in N$. By Claim 1, $\langle y_n \rangle$ has a cluster point p. For each $n \in N$, let $F_n = \{x_m : m \geqslant n\}$. Assume that $\langle x_n \rangle$ has no cluster point. Then $\langle F_n \rangle \in \mathcal{F}_0(X)$. By (1), $p \notin g(j, F_j)$ for some $j \in N$. Since p is a cluster point of $\langle y_n \rangle$, there exists $i \geqslant j$ such that $y_i \notin g(j, F_j) \supset g(i, F_i) \supset g(i, x_i)$, a contradiction.

Now, suppose that $x_n \in st^2(x, \mathcal{G}_n)$ for each $n \in N$. Then there exist $y_n, z_n, w_n \in X$ such that $x \in g(n, z_n)$, $w_n \in g(n, y_n) \cap g(n, z_n)$ and $x_n \in g(n, y_n)$ for each $n \in N$. By (2), $z_n \in g(n, x)$ and $z_n \in g(n, w_n)$ from which it follows that $g(n, x) \cap g(n, w_n) \neq \emptyset$ for all $n \in N$. By Claim 2, $\langle w_n \rangle$ has a cluster point p. Then there is a subsequence $\langle w_{n_k} \rangle$ of $\langle w_n \rangle$ such that $w_{n_k} \in g(k, p)$ for all $k \in N$. Since $w_{n_k} \in g(k, y_{n_k})$, we have that $g(k, p) \cap g(k, y_{n_k}) \neq \emptyset$ for all $k \in N$. By Claim 2,

□ynk□ has a cluster point q which is also a cluster point of □yn□. Then there is a subsequence□ymj□ of □yn□ such that ymj∈ g(j, q) for all j ∈ N. Since xn ∈ g(n, yn) for each n ∈ N, by (2),ymj∈ g(j, xmj) for each j ∈ N. It follows that g(j, q) ∩ g(j, xmj)/= ∅ for all j∈ N. By Claim 2,□xmj□ has a cluster point which is also a cluster point of □xn□. Therefore X is a wM-space.

3. Conclusions

In this section, we present characterizations of some generalized countably compact spaces such as q-spaces, quasi-Nagata spaces, quasi-γ spaces, wN-spaces, M#-spaces and wM-spaces with real-valued functions. To shorten the expressions of the corresponding results, we introduce the following notations.

Let A be a family of subsets of X, F a family of real-valued functions on X and f : A → F. For A ∈ A, we write fA instead of f(A). For a singleton {x}, we write fx instead of f{x}. Consider the following conditions.

Theorem 3.1 X is a q-space if and only if for each x ∈ X, there exists fx ∈ U(X) satisfying (c{x}) and (i{x}□Fn□).

Proof Let g be the g-function in Proposition 2.2. For each x ∈ X, let

Then fx ∈ U(X) and fx(x) = 0.

Let □Fn□ ∈ F0(X). By Proposition 2.2, there is m ∈ N such that Fm ∩ g(n, x) = ∅ for all n > m. Thus for each y ∈ Fm,

Conversely, for each x ∈ X and n ∈ N, let g(n, x) = {y ∈ X : fx(y) < 1n}. Then g(n, x)is open, x ∈ g(n, x) and g(n + 1, x) ⊂ g(n, x) which implies that g is a g-function for X. Let□Fn□ ∈ F0(X) and x ∈ X. By (i{x}□Fn□), there exists m ∈ N such that inf{fx(y) : y ∈ Fm} > 0.Then there exists k ⩾ m such that fx(y) > 1k for each y ∈ Fm. Thus for each y ∈ Fk, fx(y) > 1kwhich implies that Fk ∩ g(k, x) = ∅. By Proposition 2.2, X is a q-space.

Theorem 3.2 X is a quasi-γ space if and only if for each S ∈ SX, there exists fS ∈ U(X) satisfying (cS), (mS) and (iS□Fn□).

Proof Let g be the g-function in Proposition 2.3 (c). For each S ∈ SX, let:

,Then fS ∈ U(X) satisfies (cS) and (mS).

Theorem 3.3 X is a β-space if and only if for each F ∈ τ c, there exists fF ∈ U(X) satisfying (cF), (mF) and (i□Fn□{x}).

Proof Let g be the g-function in Proposition 2.4. Conversely, define a g-function g for X by letting g(n, x) = {y ∈ X : fx(y) < 1n} for eachx ∈ X and n ∈ N. Let □Fn□ ∈ F0(X) and x ∈ X. By (i□Fn□{x}), there exist m ∈ N and k ⩾ msuch that fFm(x) > 1k . Thus for each y ∈ Fk, fy(x) ⩾ fFk (x) ⩾ fFm(x) > 1k which implies thatx /∈g(k, Fk). By Proposition 2.4, X is a β -space.

Theorem 3.4 X is a quasi-Nagata space if and only if for each F ∈ τ c, there exists fF ∈ U(X) satisfying (cF), (mF) and (i□Fn□S) with S ∈ SX.

Conversely, define a g-function g for X by letting g(n, x) = {y ∈ X : fx(y) < 1n} for eachx ∈ X and n ∈ N. Let □Fn□ ∈ F0(X) and S ∈ SX. By (i□Fn□S), there exist m ∈ N and k ⩾ m。

such that fFm(x) > 1k for each x ∈ S. Thus for each y ∈ Fk, fy(x) ⩾ fFk (x) ⩾ fFm(x) > 1k which implies that x /∈g(k, Fk). It follows that S ∩ g(k, Fk) = ∅. By Proposition 2.5 (c), X is aquasi-Nagata space.

Theorem 3.5 X is a kβ-space if and only if for each F ∈ τ c, there exists fF ∈ U(X) satisfying (cF), (mF) and (i□Fn□K) with K ∈ CX.

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