Journal of Physics Through Computation (2019) Vol. 2: 1-5
Clausius Scientific Press, Canada

DOI: 10.23977/jptc.2019.21001
ISSN 2617-1163

A Class of Third-Order Compact Upwind Schemes for Compressible Flow with Shocks and Vortices

Abhishek Kundu[1,a,*], Arnab Basu[2,b]

[1]*Department of Mechanical Engineering, Institute of Engineering & Management, Kol-700091*
[2]*Department of Basic Science & Humanities, Institute of Engineering & Management, Kol-700091*
a. abhishek.kundu@iemcal.com, b. arnab.basu@iemcal.com
**corresponding author*

Keywords: Compact upwind scheme, CUD3, MUSCL, Euler equations

Abstract: We present a class of third-order upwind compact schemes and develop an implementation strategy for un-steady compressible flow applications with strong shocks and vortices. Two adjustable parameters are left for the user to tune the scheme for specific applications. A third-order upwind scheme reported by Tolstykh and termed CUD-3 (High accuracy non-central compact difference schemes for fluid dynamic applications, World Scientific) turns out to be a particular case of the presented class of schemes. The aforementioned adjustable parameters allow us to derive schemes that perform better than the CUD-3. A large number of compressible solvers in use today for applications involving sharp gradients in the flow field employs flux limiters. We pose our compact scheme as a cell-face variable interpolation scheme so that available limiters may be applied during construction of the cell-face fluxes. Solvers based on popular schemes such as the AUSM class of schemes or the CUSP will require minimal modification to incorporate the proposed flux reconstruction method using compact upwinding. A number of test cases in one dimensions are then solved to show the usefulness of the proposed scheme and its implementation strategy.

1. Introduction

For fluid dynamic applications, compact schemes were suggested by Tolstykh in 1972 *[1]*. The area of application was ‘the slightly rarefied hypersonic flow around blunted configurations.’ Compact schemes were also proposed by Kreiss. Its development and use in viscous test problems has been reported in *[2]*. The popularity of compact schemes grew with the publication of the paper by Lele

[3]. In a compact scheme, derivatives of a variable in several adjacent grid cells along any spatial direction are related to the values of the variable in a number of cells so that the resulting derivative can have high-order and/or high spectral accuracy. The high formal order of accuracy is accompanied by a high accuracy in the wavenumber space, but it has been shown [3] that it is possible to derive schemes with enhanced accuracy in wavenumber space for a given order of accuracy high spectral accuracy. The high formal order of accuracy is accompanied by a high accuracy in the wavenumber space, but it has been shown *[3]* that it is possible to derive schemes

with enhanced accuracy in wavenumber space for a given order of accuracy.

In incompressible flow applications, application of compact schemes is straightforward. One can either use a central scheme and add numerical diffusion from an explicit even derivative, or, can use an upwind compact scheme with inbuilt diffusion. The situation is differ-ent when shocks are present, as in high Mach number compressible flow problems. Here, across shocks one has to switch back to narrow-stencil dissipative formu-lae to avoid generation of wiggles. This is easier said than done, and has given rise to quite a few different approaches. In *[1]*, the author indicates that the wiggles, in many practical cases, are ‘well localized near discon-tinuities and do not spoil numerical solutions.’ However, in high Mach number flows, the author mentions that the third-order compact upwind scheme (CUD-3) failed to operate. As a solution, the author suggests the use of flux correction ‘introduced by Boris and Book *[4]* and further developed by Zalesak *[5]*.’ Another possible alternative is to apply filters to damp the under-resolved high wavenumber components. Developments along this line can be found in *[6]*, *[7]*. Application of filters has largely been limited to low Mach number applications. A third direction - the essentially non-oscillatory (ENO) and weighted ENO (WENO) schemes *[8]* - have found widespread use in shock and vortex dominated flows. They employ one of a number of candidate stencils to construct high-order fluxes at the cell-faces (ENO), or use a combination of all of the possible stencils (WENO). These schemes are fairly complex, and some of their variants have been found to be too dissipative in smooth regions *[9]*. A combination of ENO/WENO *[10]* schemes with compact schemes appears to be an obvious remedy of this issue, where shock capturing capacity of the former.

The paper is organized as follows. We first derive the third-order compact upwind scheme, and proceed to develop the equivalent interpolation scheme. Next, we calculate two one-dimensional problems. Finally, concluding remarks are given.

2. The Compact Scheme

We restrict ourselves to a three-cell stencil and write the compact scheme for first derivative as:

$$b_{-1}\, u'_{j-1} + b_0\, u'_j + b_1 u'_{j+1} = (a_{-1}u_{j-1}+a_0 u_j + a_1 u_{j+1})/dx \tag{1}$$

After a Taylor series expansion, and equating coefficients of derivatives upto fourth order, we arrive at the following relation among the coefficients.

$$a_{-1} = -(2\alpha - 4\beta + 3b_0)/4$$

$$a_0 = \alpha - 2\beta$$

$$a_1 = (-2\alpha - 4\beta + 3b_0)/4$$

$$b_{-1} = (\alpha - \beta + b_0)/4$$

$$b_1 = (-\alpha + \beta + b_0)/4$$

The class of third-order compact upwind scheme, pre-sented in the above form, will be termed COMPUS. In the above, β is the coefficient associated with the second derivative term $u^2\, dx/2$ collected on the right hand side of Eq. 1. We recommend α in the range of $0 < \alpha < 1{:}0$, though higher values are possible and may be explored if needed. We set $b_0 = 1$ in our applications.

3. Wave Propagation in one-dimension

The class of third-order accurate compact schemes derived in this paper has been analyzed in the previous section in the wavenumber space. We have particularly noted that greater the value of, the higher is the numerical dissipation, together with lower phase error. Since reduced phase error and reduced dissipation do not happen together, we have to carefully select the parameters and b_0 for any particular application. In the following two wave propagation problems, we show that if the wavenumber of the wave packet being propagated falls within the zone where phase error is very low, we should choose the scheme with as low dissipation as pos-sible for its time integration. We first solve the problem of Trefethen. Here we integrate the 1D wave equation for a very short time interval, and show that lower dissipation still has a significant impact on the correct amplitude of the travelling wave. The next problem is a long time integration problem taken from Bose et al.. In all these problems, time stepping is performed by a third-order accurate four-stage Runge-Kutta scheme.

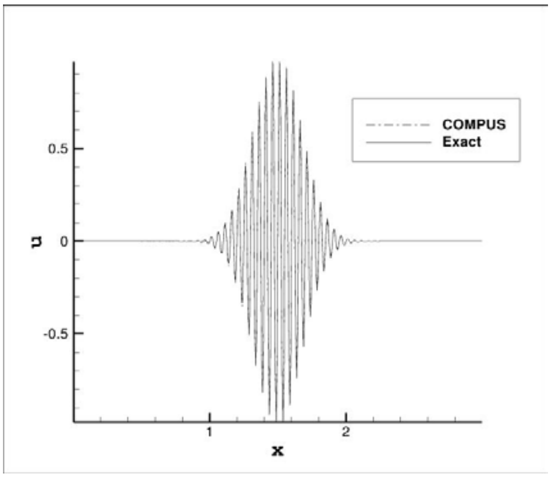

Figure 1 Wave propagation problem of Trefethen COMPUS with α= 0:01 and exact solution at t = 1.

3.1. The Problem of Trefethen

We solve the one-dimensional wave equation:

$$du/dt + c\, du/dx = 0 \tag{2}$$

We take c = 1. The domain is 0<x<3. The wave packet at t = 0 is centred at x = 0.5 and is given by

$$u(x; t = 0) = e^{-16(x-0.5)2} \sin k_0 x \tag{3}$$

There are 480 cells in the domain, so that dx = 1/160. k_0 is chosen as $k_{0\ dx = 2\pi/8,\ \text{with}}$ 8 grid cells per wavelength, suggested in *[11]*. A CFL of 0.1 is used for time step calculation. Fig. 1 shows the position of the exact solution and the numerical result of COMPUS with α= 0:01; b_0 = 1 at t = 1. We note that they are almost indistinguishable. Fig. 2 displays the results at the same time instant for (1) α = 1:0; b_0 = 1; (2) α= 0:1; b_0 = 1 (top row) and (3) α= 1:0; b_0 = 8/12 ; (4) third-order explicit MUSCL scheme (bottom row). The value of $k_0$dx is within the zone where the real part of the modified wave number is close to 1 for all the schemes. For MUSCL, it is actually at the edge of its resolving capacity. The amplitude of the wave packet is thus more for the scheme with less numerical dissipation. This figure provides evidence of how important the numerical dissipation properties of the scheme could be - a resolved wave could be almost reduced to non-existence in a short time interval if care is not exercised in scheme selection.

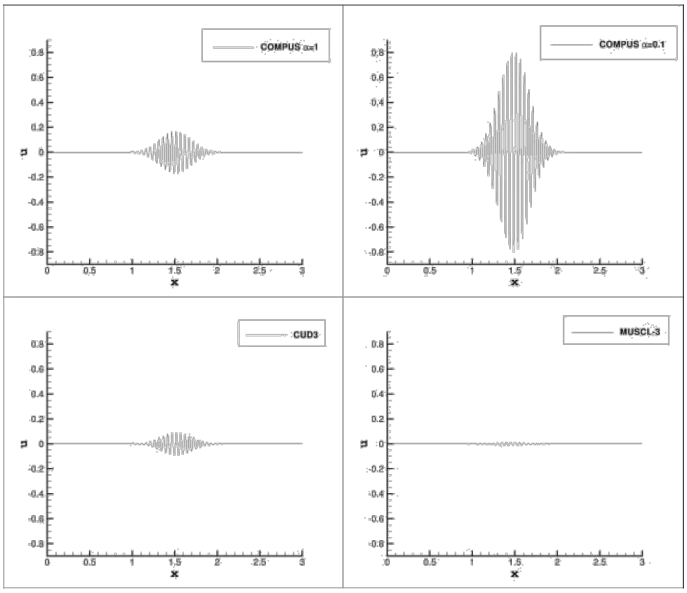

Figure 2: Wave propagation problem of Trefethen at t = 1. Top row: COMPUS with α= 1 (left frame), 0.1(right frame); b_0 = 1 for both. Bottom row: CUD-3 (left frame) and third order explicit MUSCL.

3.2. A Long Time Integration Problem

Having seen the effects of numerical dissipation in damping wave packets in a short time interval, we now observe the performance of a low-dissipation version of COMPUS in solving the same wave propagation equation over a relatively longer time interval. For this purpose we take the initial condition from Bose et al.:

$$u(x; t = 0) = e^{-20x2} \cos k_0 x \tag{4}$$

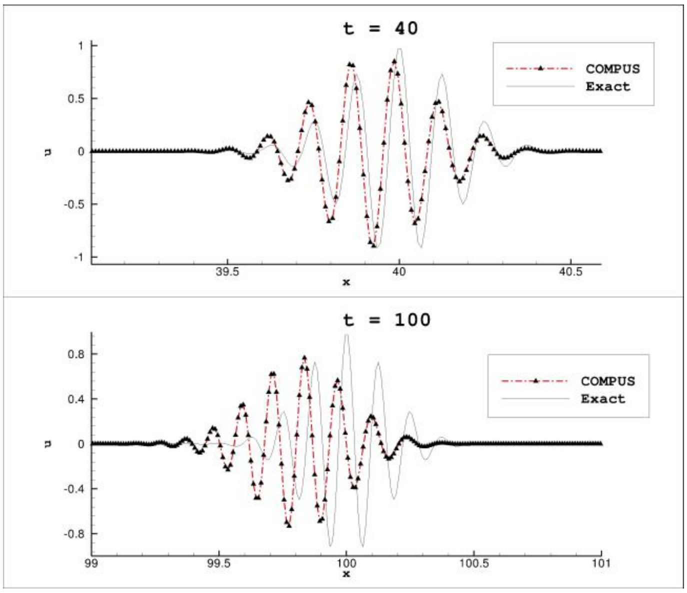

Figure 3 Long time integration of the wave equation. Computed with α = 0:01; b_0 = 1.

The computational domain extends from x = 10 to x = 200. 21000 cells have been taken in this interval to maintain x = 0:01. The authors compare the performance of several schemes for this problem, including the 4P3Om1 and 7P7Om2 schemes, the OUCS3 and the ADB scheme developed by the authors. Out of these, the first is a third-order spectrally optimized scheme involving four cells. In Fig. 3, we show the performance of the COMPUS with α= 0:01; b_0 = 1 for this problem at t = 40; 100. We have computed with the same CFL of 0.1. This corresponds to figure 10 and 11 of Bose et al. We note that the low-dissipation version of the COMPUS does much better compared to the 4P3Om1 scheme, and its performance is very close to that of the OUCS3, another optimized scheme with a larger stencil. We also notice that the 7 point optimized seventh-order scheme is unstable and produces much higher amplitude than the initial wave packet. At t = 40 the 4P3Om1 shows small ripples in the place of the actual wave packet. At t = 100, the waves are almost non-existent under this scheme. The performance of the low-dissipation COMPUS, on the other hand, is quite satisfactory.

4. Conclusion

A class of third-order compact upwind scheme has been implemented for application of unsteady compressible flow problem with shocks and vortices. Number of one-dimensional problems has been solved using this scheme. Results are matches well with the exact result. Future scope lies in the application of these scheme to two-dimensional compressible problem.

References

[1] Tolstykh, A.I. (1994) High-accuracy non-centered compact difference schemes for fluid dynamics applications. World Scientific.

[2] Hirsh, R.S. (1975) High order accurate difference solutions of fluid mechanics problems by a compact differencing technique. J. Comput. Phys., 19, 90-109.

[3] Lele, S. (1992) Compact finite difference schemes with spectral-like resolution. J. Comput. Phys., 103, 16-42.

[4] Boris, J.P., Book D.L. (1973) Flux corrected transport. I. SHASTA, a fluid transport algorithm that works. J. Comput. Phys., 11, 38-69.

[5] Zalesak, S.T. (1979) Fully multidimensional flux-corrected transport algorithms for fluids, J. Comput. Phys., 31, 335-362.

[6] Nagarajan, S., Lele, S., Ferziger, J.H. (2003) A robust high-order compact method for large eddy simulation. J. Comput. Phys., 191, 392-419.

[7] Rizzetta, D.P., Visbal, M.R., Morgan, P.E. (2008) A high-order com-pact finite-difference scheme for large-eddy simulation of active flow control. Prog. Aerosp. Sci., 44, 397-426.

[8] Jiang, G.-S., Shu, C.-W. (1996) Efficient implementation of weighted ENO schemes. J. Comput. Phys., 126, 202-228.

[9] Guo-Hua, T., Xiang-Jiang, Y. (2007) A characteristic-based shock-capturing scheme for hyperbolic problems. J. Comput. Phys., 225, 2083-2097.

[10] Pirozzoli, S (2002) Conservative-hybrid compact-WENO schemes for shock-turbulence interaction. J. Comput. Phys., 178, 81-117.

[11] Trefethen, L.N. (1982) Group velocity in finite difference schemes. SIAM Rev., 24(2), 113-136.